MAT 132A Homework assignment #4

Exercise 4.10

(i) $\{0,1,2\}$ recurrent.

(ii) $\{0,1,2,3\}$ recurrent.

(iii) $\{0,2\}$ recurrent, $\{1\}$ transient, $\{3,4\}$ recurrent.

(iv) $\{0,1\}$ recurrent, $\{2\}$ recurrent, $\{3\}$ transient, $\{4\}$ transient.

Exercise 4.13

$\sum_{i=1}^{n} Y_j/n \to E[Y]$ by the Strong Law of large numbers. Now $E[Y] = 2p - 1$. Hence, if $p > 1/2$ then $E[Y] > 0$, and so the average of the $Y_j's$ converges in this case to a positive number which implies that $\sum_1^n Y_j \to \infty$ as $n \to \infty$. Hence, state 0 can be visited only a finite number of times and so must be transient. Similarly, if $p < 1/2$, then $E[Y] < 0$, and so $\lim_{n\to\infty} \sum_1^n Y_j = -\infty$, and the argument is similar.

Exercise 4.16

If $\sum_{i=0}^{m} P_{ij} = 1$ for all j, then $\pi_j = 1/(M+1)$ satisfies

$$\pi_j = \sum_{i=0}^{m} \pi_i P_{ij}, \qquad \sum_{j=0}^{m} \pi_j = 1$$

Exercise 4.18

Let X_n denote the value of Y_n mod 13. That is, X_n is the remainder when Y_n is divided by 13. Now X_n is a Markov chain with states $0, 1, \ldots 12$. It is easy to verify that $\sum_i P + ij = 1$ for all j. (for instance, for $j = 3$:

$$\begin{aligned}\sum_i P_{ij} &= P_{2,3} + P_{1,3} + P_{0,3} + P_{12,3} + P_{11,3} + P_{10,3} \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1).\end{aligned}$$

Hence from Exercise 15, $\pi_i = \frac{1}{14}$.

Exercise 4.21

Letting X_n denote the number of pairs of shoes at the door the runner departs from at the beginning of day n, then $\{X_n\}$ is a Markov chain with transition probabilities

$P_{i,j} = \frac{1}{4}$	This is the situation in which the runner returns to the same door and then chooses that door the next day,
$P_{i,k-j} = \frac{1}{4}$	returns to the same door and then choses the other door
$P_{i,k-j} = \frac{1}{4}$	returns to the opposite door and chooses that door next
$P_{0,k} = \frac{1}{4}$	same as above
$P_{i,j-1} = \frac{1}{4}, j > 0$	returns to opposite door but selects original door next
$P_{0,0} = \frac{1}{4}$	same as above

Now, for $j \neq k$

$$\sum_i P_{ij} = P_{jj} + P_{k-j,j} + P_{k-j+1,j} + P_{j+1,j} = 1$$

Since a similar result is true for $j = k$, we see that this Markov chain is doubly stochastic; that is, all of the column sums equal 1, and thus the limiting probabilities are equal. Hence, the proportion of days he runs barefooted is $1/(k+1)$.

Exercise 4.25

Each employee moves according to a Markov chain whose limiting probabilities are the solutions of

$$\begin{aligned}\Pi_1 &= .7\,\Pi_1 + .2\,\Pi_2 + .1\,\Pi_3 \\ \Pi_2 &= .2\,\Pi_1 + .6\,\Pi_2 + .4\,\Pi_3 \\ \Pi_3 &= 1 - \Pi_1 - \Pi_2\end{aligned}$$

Solving yields $\Pi_1 = 6/17, Pi_2 = 7/17, Pi_3 = 4/17$. Hence, if N is large, it follows from the law of large numbers that approximately 6, 7, and 4 of each 17 employees are in categories 1, 2, and 3.

Exercise 4.29

Consider the Markov chain whose state at time n is the type of exam number n. The transition probabilities of this Markov chain are obtained by conditioning on the performance of the class. This gives the following.

$$\begin{aligned}P_{11} &= .3(1/3) + .7(1) = .8 \qquad & P_{12} = P_{13} = .3(1/3) = .1 \\ P_{21} &= .6(1/3) + .4(1) = .6 \qquad & P_{22} = P_{23} = .6(1/3) = .2 \\ P_{31} &= .9(1/3) + .1(1) = .4 \qquad & P_{32} = P_{33} = .9(1/3) = .3\end{aligned}$$

Let r_i denote the proportion of examrs that are type $i, i = 1, 2, 3$. The r_i are the solutions of the following set of linear equations:

$$\begin{aligned} r_1 &= .8\,r_1 + .6\,r_2 + .4\,r_3 \\ r_2 &= .1\,r_1 + .2\,r_2 + .3\,r_3 \\ r_3 &= 1 - r_1 - r_2 \end{aligned}$$

Since $P_{i2} = P_{i3}$ for all states i it follows that $r_2 = r_3$. Solving the equations gives the solution

$$r_1 = 5/7, r_2 = r_3 = 1/7$$

Exercise 4.31

The equations are

$$\begin{aligned} r_0 &= r_1 + 1/2\,r_2 + 1/3\,r_3 + 1/4\,r_4 \\ r_1 &= 1/2\,r_2 + 1/3\,r_3 + 1/4\,r_4 \\ r_2 &= 1/3\,r_3 + 1/4\,r_4 \\ r_3 &= 1/4\,r_4 \\ r_4 &= r_0 \\ r_0 &= 1 - r_1 - r_2 - r_3 - r_4 \end{aligned}$$

The solutions is

$$r_0 = r_4 = 12/37, r_1 = 6/37, r_2 = 4/37, r_3 = 3/37$$