MAT 132A Homework assignment #5

Exercise 4.36

(a) 1, since all states communicate and thus all are recurrent since state space is finite.

(b) Condition on the first state visited from i.

$$x_j = \sum_{j=1}^{N-1} P_{ij} x_j + P_{iN}, \quad i = 1, \dots, N-1$$

$$X_0 = 0, x_N = 1$$

(c) Must show:

$$\begin{aligned} \frac{i}{N} &= \sum_{j=1}^{N-1} \frac{j}{N} P_{ij} + P_{iN} \\ &= \sum_{j=0}^{N} \frac{j}{N} P_{ij} \end{aligned}$$

and it follows by hypothesis.

Exercise 4.37

(i) Let the state be the number of umbrellas he has at his present location. The transition probabilities are:

$$P_{0,r} = 1, P_{i,r-i} = 1 - p, P_{i,r-i+1} = p, i = 1, \dots, r$$

(ii) We must show that $\Pi_j = \sum_i r_i P_{ij}$ is satisfied by the given solution. These equations reduce to:

$$\begin{aligned}
\tau_r &= \tau_0 + \tau_1 p \\
\tau_j &= \tau_{r-j}(1-p) + \tau_{r-j+1} p, j = 1, \ldots, r-1 \\
\tau_0 &= \tau_r(1-p)
\end{aligned}$$

and it is easily verified that they are satisfied.

(iii)

$$p\, r_0 = \frac{p\, q}{r+q}$$

(iv)

$$\begin{aligned}
\frac{d}{dp}\left(\frac{p(1-p)}{4-p}\right) &= \frac{(4-p)(1-2p)+p(1-p)}{(4-p)^2} \\
&= \frac{p^2-8p+4}{(4-p)^2}
\end{aligned}$$

$p^2 - 8p + 4 = 0 \Rightarrow p = \frac{8-\sqrt{48}}{2} \approx .55$

Exercise 4.40

(i) No.

$$\lim P\{X_n = i\} = p\,\tau^1(i) + (1-p)\tau^2(i)$$

(ii) Yes.

$$P_{ij} = p\,P_{ij}^{(1)} + (1-p)P_{ij}^{(2)}$$

Exercise 4.44

$$\begin{aligned}
S_{11} &= P\{\text{offspring is a aa — both parents dominant}\} \\
&= \frac{r^2\frac{1}{4}}{(1-q)^2} = \frac{r^2}{4(1-q)^2} \\
S_{10} &= \frac{P\{\text{aa, 1 dominant and 1 recessive parent}\}}{P\{\text{1 dominant and 1 recessive parent}\}} \\
&= \frac{P\{\text{aa, 1 dominant and 1 recessive parent}\}}{2q(1-q)} \\
&= \frac{2qr\frac{1}{2}}{2q(1-q)} \\
&= \frac{r}{2(1-q)}
\end{aligned}$$

Exercise 4.45

This is just the probability that a gambler starting with m reaches her goal of $n+m$ before going brke, and is thus equal to $\frac{1-(q/p)^m}{1-(q/p)^{n+m}}$, where $q = 1-p$.

Exercise 4.46

Let A be the event that all states have been visited by time T. Then, comditioning on the direction of the first step gives

$$\begin{aligned}
P(A) &= P(A \text{ — clockwise})p + P(A \text{ — counterclockwise})q \\
&= p\frac{1-q/p}{1-(q/p)^n} + q\frac{1-p/q}{1-(p/q)^n}
\end{aligned}$$

The conditional probabilities in the preceding follow by noting that they are equal to the probability in the gambler's ruin problem that a gambler that starts with 1 wil reach n before going broke when the gambler's win probabilities are p and q.

Exercise 4.47

Using the hint, we see that the desired probability is:

$$\frac{P\{X_{n+1} = i+1 | X_n = i\} P\{\lim X_n = N | X_n = i, X_{n+1} = i+1\}}{P\{\lim X_n = N | X_n = i\}} = \frac{p^{P_i} + 1}{P_i}$$

and the result follows from Equation (5.1).

Exercise 4.50

(a) and (b) With $P_0 = 0, P_N = 1$

$$P_i = \alpha_i P_{i+1} + (1 - \alpha_i) P_{i-1}, \quad i = 1, \ldots, N-1$$

These latter equations can be rewritten as

$$P_{i+1} - P_i = \beta_i (P_i - P_{i-1})$$

where $\beta_i = (1 - \alpha_i)/\alpha_i$. These equations can now be solved exactly as in the original gambler's ruin problem. They give the solution:

$$P_i = \frac{1 + \sum_{j=1}^{i-1} C_j}{1 + \sum_{j=1}^{N-1} C_j}, \quad i = 1, \ldots, N-1$$

where

$$C_j = \prod_{i=1}^{j} \beta_i$$

(c) P_{N-i}, where $\alpha_i = (N-i)/N$.