MAT 132A Homework assignment #6

Exercise 4.54

$\pi \geq 0 = P(X_0 = 0)$. Assume that $\pi \geq P(X_{n-1} = 0)$.

$$
\begin{aligned}
P(X_n = 0) &= \sum_j P(X_n = 0 | X_1 = 1) P_j \\
&= \sum_j [P(X_{n-1} =)]^j P_j \\
&\leq \sum_j \pi^j P_j \\
&= \pi
\end{aligned}
$$

Exercise 4.55

(a) $\pi_0 = \frac{1}{3}$

(b) $\pi_0 = 1$

(c) $\pi_0 = (\sqrt{3} - 1)/2$

Exercise 4.56

(a) Yes, the next state depends only on the present one and not on the past.

(b) One calss period is 1, recurrent.

(c)

$$\begin{aligned}
P_{i,i+1} &= P\frac{N-i}{N}, \quad i = 0, 1, \dots, N-1 \\
P_{i,i-1} &= (1-P)\frac{i}{N}, \quad i = 1, 2, \dots, N \\
P_{i,i} &= P\frac{i}{N} + (1-P)\frac{N-i}{N}, \quad i = 0, 1, \dots, N
\end{aligned}$$

(d) See (e)

(e) $\pi_i = \binom{N}{i} p^i (1-p)^{N-i}, \quad i = 0, 1, \dots, N$

(f) Direct substitution or use Example 7a.

(g) Time $= \sum_{j=i}^{N-1} T_j$, where T_j is the number of flips to go from j to $j+1$ heads. T_j is geometric with $E(T_j) = N/j$. Thus, E(Time) $= \sum_{j=i}^{N-1} N/j$.

Exercise 4.59

(a)

$$P_{i,i+1} = \frac{(m-i)^2}{m^2}, \quad P_{i,i-1} = \frac{i^2}{m^2}, \quad P_{i,i} = \frac{2i(m-i)}{m^2}$$

(b) Since, in the limit, the set of m balls in urn 1 is equally likely to be any subset of m balls, it is intuitively clear that

$$\pi_i = \frac{\binom{m}{i}\binom{m}{m-i}}{\binom{2m}{m}} = \frac{\binom{m}{i}^2}{\binom{2m}{m}}$$

(c) We must verify that, with the π_i given in (b),

$$\pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i}$$

That is we must verify that

$$(m-i)\binom{m}{i} = (i+1)\binom{m}{i+1}$$

which is immediate.

Exercise 4.60

If $\pi = c\frac{P_{ij}}{P_{ji}}$ then

$$\pi_j P_{jk} = c\frac{P_{ij}P_{jk}}{P_{ji}}$$

$$\pi_k P_{kj} = c\frac{P_{ik}P_{kj}}{P_{ki}}$$

and are thus equal by hypothesis.

Exercise 4.63

(a) The state would be the present ordering of the n processors. Thus, there are $n!$ states.

(b) Consider states $x = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ and $x' = (x_1, \ldots, x_{i-1}, x_{i+1}, x_i, \ldots, x_n)$. With q_t equal to $1 - p_t$ the time reversible equations are

$$\pi(x) q_{x_i} p_{x_{i+1}} \prod_{k=1}^{i-1} q_{x_k} = \pi(x') q_{x_{i+1}} p_{x_i} \prod_{k=1}^{i-1} q_{x_k}$$

or

$$\pi(x) = (q_{x_{i+1}}/p_{x_{i+1}})(q_{x_i}/p_{x_i})^{-1}\pi(x')$$

Suppose now that we successively utilize the above identity until we reach the state $(1, 2, \dots, n)$. Note that each time j is moved to the left we multiply by q_j/p_j and each time it moves to the right we multiply by $(q_j/p_j)^{-1}$. Since x_j, which is initially in position j, is to have a net move of $j - x_j$ positions to the left (so it will end up in position $j - (j - x_j) = x_j$) it follows from the above that

$$\pi(x) = C \prod_j (q_{x_j}/p_{x_j})^{j-x_j}$$

The value of C, which is equal to $\pi(1, 2, \dots, n)$, can be obtained by summing over all states x and equating to 1. Since the solution given by the above value of $\pi(x)$ satisfies the time reversibility equations, it follows that the chain is time reversible and these are the limiting probabilities.

Exercise 4.64

The number of transitions from i to j on any interval must be equal (to within 1) the number from j to i since each time the process goes from i to j in order to get back to i, it must enter from j.