MAT 132A Homework assignment # 8

Exercise 4.66

(a)

$$\begin{aligned}
\sum_a y_{ja} &= \sum_a E_\beta \left[\sum \alpha^n I_{\{X_n=j, a_n=a\}}\right] \\
&= E_\beta \left[\sum_n \alpha^n \sum_a I_{\{X_n=j, a_n=a\}}\right] \\
&= E_\beta \left[\sum_n \alpha^n I_{\{X_n=j\}}\right]
\end{aligned}$$

(b)

$$\begin{aligned}
\sum_j \sum_a y_{ja} &= E_\beta \left[\sum_n \alpha^n \sum_j I_{\{X_n=j\}}\right] \\
&= E_\beta \left[\sum \alpha^n\right] = \frac{1}{1-\alpha}
\end{aligned}$$

$$
\begin{aligned}
\sum_a y_{ja} &= b_j + E_\beta\left[\sum_{n=1}^{\infty} \alpha^n I_{\{X_n=j\}}\right] \\
&= b_j + E_\beta\left[\sum_{n=0}^{\infty} \alpha^{n+1} I_{\{X_{n+1}=j\}}\right] \\
&= b_j + E_\beta\left[\sum_{n=0}^{\infty} \alpha^{n+1} \sum_{i,a} I_{\{X_n=i,a_n=a\}} I_{\{X_{n+1}=j\}}\right] \\
&= b_j + \sum_{n=0}^{\infty} \alpha^{n+1} \sum_{i,a} E_\beta\left[I_{\{X_n=i,a_n=a\}}\right] P_{ij}(a) \\
&= b_j + \alpha \sum_{i,a} \sum_{n} \alpha^n E_\beta\left[I_{\{X_n=i,a_n=a\}}\right] P_{ij}(a) \\
&= b_j + \alpha \sum_{i,a} y_{ia} P_{ij}(a)
\end{aligned}
$$

(c) Let $d_{j,a}$ denote the expected discounted time the process is in j, and a is chosen when policy β is employed. Then by the same argument as in (b):

$$
\begin{aligned}
\sum_a d_{ja} &= b_j + \alpha \sum_{i,a} \sum_{n} \alpha^n E_\beta\left[I_{\{X_n=i,a_n=a\}}\right] P_{ij}(a) \\
&= b_j + \alpha \sum_{i,a} \sum_{n} \alpha^n E_\beta\left[I_{\{X_n=i\}}\right] \frac{y_{ia}}{\sum_a y_{ia}} P_{ij}(a) \\
&= b_j + \alpha \sum_{i,a} \sum_{a'} d_{ia'} \frac{y_{ia}}{\sum_a y_{ia}} P_{ij}(a)
\end{aligned}
$$

and we see from Equation (4.35) that the above is satified upon substitution of $d_{ia} = y_{ia}$. As it is easy to see that $\sum_{i,a} d_{ia} =$

$\frac{1}{1-\alpha}$, the result follows since it can be shown that these linear equations have a unique solution.

(d) Follows immediately from previous parts.