MAT 132A Homework assignment #9

Exercise 5.5 e^{-1} by lack of memory.

Exercise 5.7

$$\begin{aligned} p\{X_1 < X_2 | \min(X_1, X_2) = t\} &= \frac{P\{X_1 < X_2, \min(X_1, X_2 = t)\}}{P\{\min(X_1, X_2) = t\}} \\ &= \frac{P\{X_1 = t, X_2 > t\}}{P\{X_1 = t, X_2 > t\} + P\{X_2 = t, X_1 > t\}} \\ &= \frac{f_1(t)[1 - F_2(t)]}{f_1(t)[1 - F_2(t)] + f_2(t)[1 - F_1(t)]} \end{aligned}$$

Dividing through by $[1 - F_1(t)][1 - F_2(t)]$ yields the result. Here f_i and F_i are the density and distribution function of $X_i, i = 1, 2$. To make the above derivation rigorous, we should replace " =t" by $\epsilon(t, t + \epsilon)$ throughout and then let $\epsilon \to 0$.

Exercise 5.14

(a) The conditional density of X given that $X < c$ is

$$f(x|X < c) = \frac{f(x)}{P\{X < c\}} = \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda c}}, \qquad 0 < x < c$$

hence,

$$E[X|X < c] = \int_0^c x\lambda e^{-\lambda x} dx / (1 - e^{-\lambda c})$$

Integration by parts yields that

$$\begin{aligned} \int_0^c x\lambda e^{-\lambda x} dx &= xe^{-\lambda x}|_0^c + \int_0^c e^{-\lambda x} dx \\ &= ce^{-\lambda c} + (1 - e^{-\lambda c})/\lambda \end{aligned}$$

Hence,

$$E[X|X < c] = 1/\lambda - ce^{-\lambda c}/(1 - e^{-\lambda c})$$

(b)

$$1/\lambda = E[X|X < c](1 - e^{-\lambda c}) + (c + 1/\lambda)e^{-\lambda c}$$

This simplifies to the answer given in part (a).

Exercise 5.37

(a) $196/2.5 = 78.4$

(b) $196/(2.5)^2 = 31.36$

We use the central limit theorem to justify approximating the life distribution by a normal distribution with mean 78.4 and standard deviation $\sqrt{31.36} = 5.6$. In the following, Z is a standard normal random variable.

(c) $P\{L < 67.2\} \approx P\{Z < \frac{67.2-78.4}{5.6}\} = P\{Z < -2\} = 0.0227$

(d) $P\{L > 90\} \approx P\{Z > \frac{90-78.4}{5.6}\} = P\{Z > 2.07\} = 0.0192$

(e) $P\{L > 100\} \approx P\{Z > \frac{100-78.4}{5.6}\} = P\{Z > 3.857\} = 0.00006$

Exercise 5.40

(a) $E[S_4] = 4/\lambda$

(b) $E[S_4|N(1) = 2] = 1 + E[\text{time for 2 more events}] = 1 + 2/\lambda$

(c) $E[N(4) - N(2)|N(1) = 3] = E[N(4) - N(2)] = 2/\lambda$
The first equality used the independent increments property.

Exercise 5.48

Let T denote the time until the next train arrives; and so T is uniform on $(0, 1)$. Note that, conditional on T, X is Poisson with mean $7T$.

(a) $E[X] = E[E[X|T]] = E[7T] = 7/2$

(b) $E[X|T] = 7T, \mathrm{Var}(X|T) = 7T$. By the conditional variance formula

$$\mathrm{Var}(X) = 7E[T] + 49\,\mathrm{Var}(T) = 7/2 + 49/12 = 91/12$$

Exercise 5.56

(a) $1 - e^{-3}$

(b) 67

(c) $\binom{50}{5}\left(\frac{1}{20}\right)^5\left(\frac{19}{20}\right)^{45}$

Exercise 5.59

Each of a Poisson number of events is classified as either being of type 1 (if found by proofreader 1, but not by 2) or type 2 (if found by proofreader 2, but not 1) or type 3 (if found by both). or type 4 (if found by neither).

(a) The X_i are independent Poisson random variables with means

$$E[X_1] = \lambda p_1(1-p_2), \qquad E[X_2] = \lambda(1-p_1)p_2$$
$$E[X_3] = \lambda p_1 p_2, \qquad E[X_4] = \lambda(1-p_1)(1-p_2)$$

(b) Follows from the above.