MATH 132A: Intro. to Stochastic Processes
MidTerm Exam, Friday February 16, 2001

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Problem 1 (20 pts) Let $X_1, X_2, \ldots$ be identically distributed random variables with mean μ and variance σ^2. Let N be a random variable taking values in the non-negative integers and independent of the X_i's. Let $S = X_1 + X_2 + \cdots + X_N$.

(a) (10 pts) Show that

$$E[S] = \mu E[N].$$

(b) (10 pts) The variance of S can be written as

$$\mathrm{var}[S] = \sigma^2 E[N] + \mu^2 \mathrm{var}[N]$$

Suppose now that the random variable N obeys the Poisson distribution with the parameter λ. Then, compute this variance $\mathrm{var}[S]$.

(a) $E[S] = E\left[\sum_{n=1}^{N} X_n\right] = E\left[E\left[\sum_{n=1}^{N} X_n \mid N\right]\right]$

Now $E\left[\sum_{n=1}^{N} X_n \mid N\right] = N\,E[X] = N\mu$

$E[S] = E[N\mu] = \mu E[N]$ ///

(b) N obeys the Poisson distribution with parameter λ

i.e. $P\{N = n\} = e^{-\lambda}\frac{\lambda^n}{n!}$

We know that $E[N] = \lambda$, $\mathrm{var}[N] = \lambda$.

$\mathrm{var}[S] = \sigma^2\lambda + \mu^2\lambda = \lambda(\sigma^2 + \mu^2)$

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Problem 2 (30 pts) Consider the Markov chain whose transition probability matrix is:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

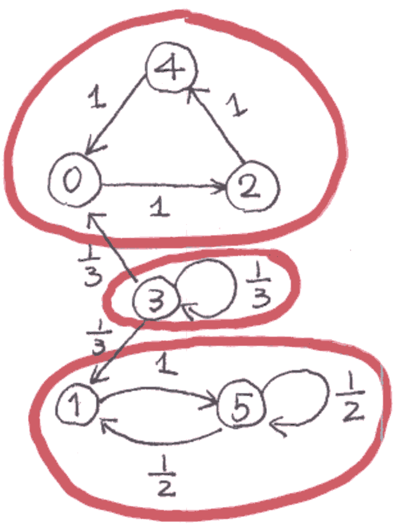
(a) (5 pts) Classify the states $\{0, 1, 2, 3, 4, 5\}$ into classes.

(b) (5 pts) Identify the recurrent and transient classes of (a).

(c) (10 pts) Compute the period of each recurrent class.

(d) (10 pts) Identify the ergodic states.

(a) The state diagram is



it is clear that we have the following three classes

$\{0, 2, 4\}$, $\{3\}$, $\{1, 5\}$. ///

(b) Again, clearly from the diagram

$\{0, 2, 4\}$: recurrent $\{3\}$: transient, $\{1, 5\}$ = recurrent. ///

(c) For {0 2, 4} class P^n 0 for ∓3 $\not\equiv 0 \pmod 3$

So the period $d = 3$ //

5} class it possible to come back to state 1

$n = 2, 3, 4$ and it possible to come back to

state 5 for 2 3

These imply that d 1 ///

(d Since this a finite state Markov chain

all ~~the~~ recurrent states are positive recurrent

~~Possi~~ Positive aperiodic (i.e d 1) states are called

ergodic

Therefore the ergodic states are 1 5} ///

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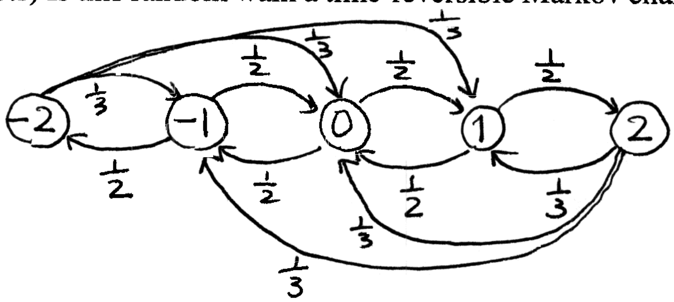
Problem 3 (30 pts) Consider a random walk on the finite states $\{-2, -1, 0, 1, 2\}$. If the process is in state i ($i = -1, 0, 1$) at time n, then it moves to either $i - 1$ or $i + 1$ at time $n + 1$ with equal probability. If the process is in state -2 or 2 at time n, then it moves to state -1, 0, or 1 at time $n + 1$ with equal probability.

(a) (10 pts) Write the transition probability matrix $\boldsymbol{P}$ for this random walk.

(b) (10 pts) Compute the stationary probability $\boldsymbol{\pi} = (\pi_{-2}, \pi_{-1}, \pi_0, \pi_1, \pi_2)$ of this random walk. [Hint: You can use the symmetry so that $\pi_{-i} = \pi_i$ for $i = 1, 2$.]

(c) (10 pts) Is this random walk a time-reversible Markov chain? State your reasoning too.

(a)



$$P = \begin{array}{c} \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccccc} -2 & & 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}$$

///

(b)

$$(\pi_{-2}\ \pi_{-1}\ \pi_0\ \pi_1\ \pi_2) = (\pi_{-2}\ \pi_{-1}\ \pi_0\ \pi_1\ \pi_2)\,P$$

from the symmetry $\pi_{-2} = \pi_2$, $\pi_{-1} = \pi_1$, we have

$$\begin{cases} \pi_0 = \frac{1}{3}\pi_2 + \frac{1}{2}\pi_1 + \frac{1}{2}\pi_1 + \frac{1}{3}\pi_2 \\ \pi_1 = \frac{1}{3}\pi_2 + \frac{1}{2}\pi_0 + \frac{1}{3}\pi_2 \\ \pi_2 = \frac{1}{2}\pi_1 \end{cases}$$

$$\pi_0 + 2\pi_1 + 2\pi_2 = 1$$

From these we can get. $\pi_0 = \frac{4}{3}\pi_1$, and $\pi_1 = 3/13$ so

$$\pi = \left(\tfrac{3}{26}\ \tfrac{3}{13}\ \tfrac{4}{13}\ \tfrac{3}{13}\ \tfrac{3}{26}\right) \quad ///$$

(c) Need to check

$$\underline{P}_{ij} \quad \pi \quad \underline{P} \qquad \text{for all}$$

But say $\quad 2 \quad j \quad 1$

$$\underline{P}_2 \qquad \frac{3}{26} \quad 3 \quad \overline{26}$$

$$\pi_1 \underline{P}_2 \qquad \frac{3}{3} \quad 2 \quad \frac{3}{26}$$

So apparently $\pi \underline{P}_{12} \neq \pi_2 \underline{P}$

This implies that this MC not time reversible.

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Problem 4 (20 pts) Consider a branching process with probabilities P_j for the number of offspring of one individual to be j, and suppose that the process starts with one individual. Calculate the probability π_0, i.e., the probability that the population dies out, in each of the following cases:

(a) (5 pts) $P_0 = 1/2, P_1 = 1/4, P_2 = 1/4$.

(b) (5 pts) $P_0 = 1/4, P_1 = 1/2, P_2 = 1/4$.

(c) (5 pts) $P_0 = 1/4, P_1 = 1/4, P_2 = 1/2$.

(d) (5 pts) $P_0 = 1/8, P_1 = 1/2, P_2 = 3/8$.

(a) $\mu = \sum_{j=0}^{2} j P_j = 0 + \cdot\frac{1}{4} + 2\cdot\frac{1}{4} = \frac{3}{4} \leq 1$

So $\pi_0 = 1$. //

(b) $\mu = 0 + 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} = 1$

So again $\pi_0 = 1$ //

(c) $\mu = 0 + 1\cdot\frac{1}{4} + 2\cdot\frac{1}{2} = 1\frac{1}{4}$

Now need to solve

$$\pi_0 = \sum_{j=0}^{2} \pi_0^j P_j = \frac{1}{4} + \frac{1}{4}\cdot$$

$$\Rightarrow \quad 2\pi_0^2 \quad 3\pi_0 + 1 = 0 \qquad (2\pi_0 - 1)(\pi_0 - 1) = 0$$

$$\pi_0 = \frac{1}{2}. \text{ or } 1.$$

Should select the ~~minimum~~ smallest positive root.

So. $\pi_0 = \frac{1}{2}$ //

(d) $\mu = 0 + 1\cdot\frac{1}{2} + 2\cdot\frac{3}{8} = \frac{5}{4} > 1$

$$\pi_0 = \sum_{0}^{2} \pi_0^j P_j = \frac{1}{8} + \frac{1}{2}\pi_0 + \frac{3}{8}\pi_0^2$$

$$3\pi_0^2 - 4\pi_0 + \; = 0 \qquad (3\pi_0 - 1)(\pi_0 - 1) = 0$$

$$\pi_0 = \frac{1}{3}$$ ///