MAT 167 Practice Midterm Exam

The actual exam will be conducted in class on Wednesday Nov. 8, 2006

Name: ______________________________

Student ID #: ______________________________

Name of Left Neighbor: ______________________________

Name of Right Neighbor: ______________________________

If you are next to the aisle, then please write "aisle" appropriately as your left or right neighbor.

- Read each problem carefully.
- Write every step of your reasoning clearly.
- Usually, a better strategy is to solve the easiest problem first.
- This is a closed-book exam. You may not use the textbook, crib sheets, notes, or any other outside material. Do not bring your own scratch paper. Do not bring blue books.
- No calculators/laptop computers/cell phones are allowed for the exam. The exam is to test your basic understanding of the material.
- Everyone works on their own exams. Any suspicions of collaboration, copying, or otherwise violating the Student Code of Conduct will be forwarded to the Student Judicial Board.

Problem #	Score
1 (20 pts)	
2 (20 pts)	
3 (20 pts)	
4 (20 pts)	
5 (20 pts)	
Total	

Problem 1 (20 pts) Suppose we are given three data points in $\mathbb{R}^2$, $(x, y) = (-\pi/2, 1), (0, 1), (\pi/2, -1)$. Now, we want to find the best function to fit these points in the form of $y = \alpha + \beta \sin x$ in the sense of the least squares.

(a) (7 pts) Write a system of equation in the form $A\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{x} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ as if this function passes through all these three points.

(b) (7 pts) Solve the least squares problem using the normal equation, and write the solution in the form of $y = \alpha + \beta \sin x$.

(c) (6 pts) Let $\hat{\boldsymbol{x}}$ be the least squares solution you computed in (b). Compute the residual (or error) vector $\boldsymbol{b} - A\hat{\boldsymbol{x}}$ and its length in 2-norm. Compare this error size o the error size of the case $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and confirm that the error of the least squares solution is smaller.

Problem 2 (20 pts) Consider the following matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

(a) (5 pts) Find a basis of $\mathcal{R}(A)$. What is the rank of this matrix?

(b) (5 pts) What is $\mathcal{N}(A)$ in this case?

(c) (5 pts) Find a basis of $\mathcal{R}(A^T)$.

(d) (5 pts) Find a basis of $\mathcal{N}(A^T)$.

Problem 3 (20 pts) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transform defined by $T(x,y) = (2x+y, x+2y)$. Consider the vector $\mathbf{v} = \begin{bmatrix}1\\2\end{bmatrix}$ and the basis

$$\mathcal{B} = \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix} \right\}.$$

Let $U = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}$ be the matrix representing this basis.

(a) (5 pts) Determine $[T]_{\mathcal{B}}$ and $[\mathbf{v}]_{\mathcal{B}}$.

(b) (5 pts) Compute $[T(\mathbf{v})]_{\mathcal{B}}$ and verify that $[T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{B}}$.

(c) (5 pts) Now, let a new basis in $\mathbb{R}^2$ be

$$\widetilde{\mathcal{B}} = \left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix} \right\}.$$

Let $V = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$ be the matrix representing this basis. Now determine the change of basis matrix $[I]_{\mathcal{B}\widetilde{\mathcal{B}}}$.

(d) (5 pts) determine $[T]_{\mathcal{B}\widetilde{\mathcal{B}}}$ and demonstrate that $[I]_{\mathcal{B}\widetilde{\mathcal{B}}}[T]_{\widetilde{\mathcal{B}}} = [T]_{\mathcal{B}\widetilde{\mathcal{B}}}$.

Problem 4 (20 pts) Consider the following matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

(a) (5 pts) Compute the Frobenius norm $\|A\|_F$.

(b) (4 pts) Compute the 1-norm $\|A\|_1$.

(c) (7 pts) Compute the 2-norm $\|A\|_2$.

(d) (4 pts) Compute the ∞-norm $\|A\|_\infty$.

Problem 5 (20 pts) Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an *orthonormal basis* of an inner product space $\mathcal{V}$ with $\dim \mathcal{V} = n$.

(a) (5pts) Let $\boldsymbol{x} \in \mathcal{V}$ be any vector in $\mathcal{V}$. Express $\boldsymbol{x}$ as a linear combination of the basis set $\mathcal{B}$, i.e., determine the linear combination coefficients, $\alpha_1, \ldots, \alpha_n$ such that $\boldsymbol{x} = \alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n$.

(b) (5 pts) Prove the Pythagorean theorem:

$$\|\boldsymbol{x}\|^2 = |\alpha_1|^2 + \cdots + |\alpha_n|^2.$$

(c) (5 pts) Suppose $n = 3$ and $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, which are not orthonormal. Make this set orthonormal.

(d) (5 pts) Expand a vector $\boldsymbol{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ with respect to the orthonormal vectors derived in Part (c).