MAT 167 Practice Midterm Exam

The actual exam will be conducted in class on
Wednesday Nov. 8, 2006

Name: ________________________________________________
Student ID #: _________________________________________
Name of Left Neighbor: _________________________________
Name of Right Neighbor: _________________________________
If you are next to the aisle, then please write “aisle” appropriately as your left or right neighbor.

• Read each problem carefully.

• Write every step of your reasoning clearly.

• Usually, a better strategy is to solve the easiest problem first.

• This is a closed-book exam. You may not use the textbook, crib sheets, notes, or any other outside material. Do not bring your own scratch paper. Do not bring blue books.

• No calculators/laptop computers/cell phones are allowed for the exam. The exam is to test your basic understanding of the material.

• Everyone works on their own exams. Any suspicions of collaboration, copying, or otherwise violating the Student Code of Conduct will be forwarded to the Student Judicial Board.

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<th>Problem #</th>
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<tr>
<td>1 (20 pts)</td>
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Problem 1 (20 pts) Suppose we are given three data points in $\mathbb{R}^2$, \((x, y) = (\frac{-\pi}{2}, 1), (0, 1), (\frac{\pi}{2}, -1)\).

Now, we want to find the best function to fit these points in the form of \(y = \alpha + \beta \sin x\) in the sense of the least squares.

(a) (7 pts) Write a system of equation in the form \(Ax = b\), where \(x = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}\) as if this function passes through all these three points.

(b) (7 pts) Solve the least squares problem using the normal equation, and write the solution in the form of \(y = \alpha + \beta \sin x\).

(c) (6 pts) Let $\hat{x}$ be the least squares solution you computed in (b). Compute the residual (or error) vector \(b - A\hat{x}\) and its length in 2-norm. Compare this error size with the error size of the case \(\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\), and confirm that the error of the least squares solution is smaller.
Problem 2 (20 pts) Consider the following matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}.$$  

(a) (5 pts) Find a basis of $\mathbb{R}(A)$. What is the rank of this matrix?

(b) (5 pts) What is $N(A)$ in this case?

(c) (5 pts) Find a basis of $\mathbb{R}(A^T)$.

(d) (5 pts) Find a basis of $N(A^T)$. 
Problem 3 (20 pts) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transform defined by $T(x, y) = (2x + y, x + 2y)$.

Consider the vector $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

Let $U = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ be the matrix representing this basis.

(a) (5 pts) Determine $[T]_B$ and $[v]_B$.

(b) (5 pts) Compute $[T(v)]_B$ and verify that $[T]_B[v]_B = [T(v)]_B$.

(c) (5 pts) Now, let a new basis in $\mathbb{R}^2$ be

$$\tilde{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

Let $V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ be the matrix representing this basis. Now determine the change of basis matrix $[I]_{B \tilde{B}}$.

(d) (5 pts) Determine $[T]_{B \tilde{B}}$ and demonstrate that $[I]_{B \tilde{B}}[T]_{\tilde{B}} = [T]_{B \tilde{B}}$. 
Problem 4 (20 pts) Consider the following matrix

\[
A = \begin{bmatrix}
1 & -1 \\
1 & 0 \\
1 & 1
\end{bmatrix}.
\]

(a) (5 pts) Compute the Frobenius norm \( \| A \|_F \).

(b) (4 pts) Compute the 1-norm \( \| A \|_1 \).

(c) (7 pts) Compute the 2-norm \( \| A \|_2 \).

(d) (4 pts) Compute the \( \infty \)-norm \( \| A \|_\infty \).
Problem 5 (20 pts) Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an orthonormal basis of an inner product space $\mathcal{V}$ with $\dim \mathcal{V} = n$.

(a) (5 pts) Let $\mathbf{x} \in \mathcal{V}$ be any vector in $\mathcal{V}$. Express $\mathbf{x}$ as a linear combination of the basis set $\mathcal{B}$, i.e., determine the linear combination coefficients, $\alpha_1, \ldots, \alpha_n$ such that $\mathbf{x} = \alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n$.

(b) (5 pts) Prove the Pythagorean theorem:

$$\|\mathbf{x}\|^2 = |\alpha_1|^2 + \cdots + |\alpha_n|^2.$$

(c) (5 pts) Suppose $n = 3$ and $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, which are not orthonormal. Make this set orthonormal.

(d) (5 pts) Expand a vector $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ with respect to the orthonormal vectors derived in Part (c).