Problem 1 (20 pts) We would like to find the best line for the given three points $(0,\alpha)$, $(1,1)$, $(2,2)$ in the plane $\mathbb{R}^2$ in the least squares sense, where α is some real-valued parameter.

(a) (6 pts) Let us write an equation of line as $y = px + q$. Then write a system of equation of the form $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{x} = \begin{bmatrix} q \\ p \end{bmatrix}$, as if the line passes through all the three points.

Answer: Entering $x = 0,1,2$ and $y = \alpha,1,2$, respectively to the equation of line $y = px + q$ yields the following system of equations.

$$\boxed{\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} q \\ p \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} \alpha \\ 1 \\ 2 \end{bmatrix}}_{\boldsymbol{b}}}$$

(b) (7 pts) Show that this system is inconsistent (i.e., the line cannot pass all the three points) if $\alpha \neq 0$. What happens when $\alpha = 0$?

Answer: Form the augmented matrix, and obtain its Reduced Row Echelon Form:

$$\left[\begin{array}{cc|c} 1 & 0 & \alpha \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cc|c} 1 & 0 & \alpha \\ 0 & 1 & 1-\alpha \\ 0 & 2 & 2-\alpha \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_2} \left[\begin{array}{cc|c} 1 & 0 & \alpha \\ 0 & 1 & 1-\alpha \\ 0 & 0 & \alpha \end{array}\right]$$

So, if $\alpha \neq 0$, then the $(3,3)$ element α does not match with the zero elements of the last row. Therefore, this system is inconsistent.

If $\alpha = 0$, then the system is consistent. By inserting $\alpha = 0$ to the above matrix, we get:

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right],$$

which means that the solution is $(q,p) = (0,1)$. Thus, we have the line $y = x$ passing through all these three points if $\alpha = 0$.

(c) (7 pts) Form the normal equation, and compute the best line by solving this normal equation.

Answer: This is simply can be solved by

$$\begin{bmatrix} q \\ p \end{bmatrix} = (A^T A)^{-1} A^T \boldsymbol{b} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}^{-1} \begin{bmatrix} \alpha + 3 \\ 5 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} \alpha + 3 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{5\alpha}{6} \\ 1 - \frac{\alpha}{2} \end{bmatrix}.$$

Thus the least square solution is:

$$\boxed{y = \left(1 - \frac{\alpha}{2}\right) x + \frac{5\alpha}{6}}.$$

Problem 2 (20 pts) Consider the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}.$$

(a) (5 pts) Find a basis of $\mathcal{R}(A)$.

Answer: Let's compute E_A, the RREF of A, together with the transformation matrix $P \in \mathbb{R}^{2\times 2}$ so that $PA = E_A$.

$$A = \left[\begin{array}{ccc|cc} 1 & 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|cc} 1 & 0 & 2 & 1 & 0 \\ 0 & 2 & -4 & -2 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow (1/2)R_2} \left[\begin{array}{ccc|cc} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 1/2 \end{array}\right] = E_A$$

Thus it is clear that the first two columns are the basic columns of A, which are the basis of $\mathcal{R}(A)$. In other words,

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}}.$$

(b) (5 pts) Find a basis of $\mathcal{N}(A)$.

Answer: We need to solve the homogeneous system of equations, $A\boldsymbol{x} = \mathbf{0}$. Since we already know the RREF E_A, we need to solve:

$$E_A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From this, we get the following equations:

$$\begin{cases} x + 2z &= 0 \\ y - 2z &= 0. \end{cases}$$

Thus, z is a free variable, and the general solution is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2z \\ 2z \\ z \end{bmatrix} = z \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = z \cdot \boldsymbol{h}_1.$$

Therefore, the basis of $\mathcal{N}(A)$ is

$$\boxed{\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}}.$$

(c) (5 pts) Find a basis of $\mathcal{R}(A^T)$.

Answer: The nonzero rows of E_A form a basis $\mathcal{R}(A^T)$. Therefore, the answer is:

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}}.$$

(d) (5 pts) What is $\mathcal{N}(A^T)$ in this case?

Answer: Using the Rank-Nullity Theorem for A^T, we have:

$$\dim(\mathcal{R}(A^T)) + \dim(\mathcal{N}(A^T)) = 2$$

And clearly $\dim(\mathcal{R}(A^T)) = \text{rank}(A^T) = \text{rank}(A) = 2$. Therefore, $\dim(\mathcal{N}(A^T)) = 0$, which means that

$$\boxed{\mathcal{N}(A^T) = \{\mathbf{0}\} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}}.$$

Problem 3 (20 pts) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transform defined by $T(x,y) = \left(\frac{\sqrt{3}x+y}{2}, \frac{-x+\sqrt{3}y}{2}\right)$.

Consider the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \right\}.$$

(a) (6 pts) Determine $[T]_{\mathcal{B}}$ and $[\mathbf{v}]_{\mathcal{B}}$.

Answer: Let A be the matrix associated with this linear transformation T. Then,

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}x+y}{2} \\ \frac{-x+\sqrt{3}y}{2} \end{bmatrix} \Longrightarrow A = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Let U be a matrix representing the basis $\mathcal{B}$, i.e.,

$$U = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

Now, we have

$$[T]_{\mathcal{B}} = U^{-1}AU = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \boxed{\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}}.$$

As for $[\mathbf{v}]_{\mathcal{B}}$, it is easy to get:

$$[\mathbf{v}]_{\mathcal{B}} = U^{-1}\mathbf{v} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1+\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \end{bmatrix}}.$$

(b) (7 pts) Compute $[T(\mathbf{v})]_{\mathcal{B}}$ and verify that $[T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{B}}$.

Answer: In this case,

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}+1}{2} \\ \frac{\sqrt{3}-1}{2} \end{bmatrix}.$$

So,

$$[T(\mathbf{v})]_{\mathcal{B}} = U^{-1}(A\mathbf{v}) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}+1}{2} \\ \frac{\sqrt{3}-1}{2} \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}.$$

On the other hand, using the results of Part (a), we have:

$$[T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}.$$

So, they surely agree.

(c) (7 pts) Now, let a new basis in $\mathbb{R}^2$ be

$$\widetilde{\mathcal{B}} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}.$$

Then determine $[T]_{\mathcal{B}\widetilde{\mathcal{B}}}$.

Answer: Let $\widetilde{U}$ be a matrix representing the basis $\widetilde{\mathcal{B}}$:

$$\widetilde{U} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

So, $[T]_{\mathcal{B}\widetilde{\mathcal{B}}}$ can be easily computed as follows:

$$\begin{aligned} [T]_{\mathcal{B}\widetilde{\mathcal{B}}} &= \widetilde{U}^{-1} A U \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \\ &= \frac{1}{4\sqrt{2}} \begin{bmatrix} \sqrt{3}+1 & -\sqrt{3}+1 \\ \sqrt{3}-1 & \sqrt{3}+1 \end{bmatrix} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \\ &= \boxed{\frac{1}{2\sqrt{2}} \begin{bmatrix} \sqrt{3}-1 & -\sqrt{3}-1 \\ \sqrt{3}+1 & \sqrt{3}-1 \end{bmatrix}}. \end{aligned}$$

Problem 4 (20 pts) Consider the following matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad 0 \le \theta \le \frac{\pi}{2}.$$

(a) (5 pts) Compute the Frobenius norm $\|R_\theta\|_F$.

Answer: Using the definition of the Frobenius norm, we have

$$\|R_\theta\|_F = \sqrt{2\left(\cos^2\theta + \sin^2\theta\right)} = \boxed{\sqrt{2}}.$$

(b) (5 pts) Explain why $\|R_\theta\|_2 = 1$.

Answer:

(c) (5 pts) Show that $\|R_\theta\|_1 = \|R_\theta\|_\infty$.

Answer: Using the definition of the 1-norm, we have

$$\|R_\theta\|_1 = \max_j \|\boldsymbol{R}_{\theta,*j}\|_1 = |\cos\theta| + |\sin\theta| = \cos\theta + \sin\theta \quad \text{since } 0 \le \theta \le \frac{\pi}{2}.$$

Similarly, we have:

$$\|R_\theta\|_\infty = \max_i \|\boldsymbol{R}_{\theta,i*}\|_1 = |\cos\theta| + |\sin\theta| = \cos\theta + \sin\theta.$$

Hence, in this case, we have

$$\boxed{\|R_\theta\|_1 = \|R_\theta\|_\infty}.$$

(d) (5 pts) For what value of θ is $\|R_\theta\|_1$ maximized? What is that maximum value of $\|R_\theta\|_1$?

Answer: The easiest way to solve this problem is to use the trigonometric identity, i.e.,

$$\|R_\theta\|_1 = \cos\theta + \sin\theta = \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right).$$

Because $0 \le \theta \le \frac{\pi}{2}$, this quantity attains the maximum at $\boxed{\theta = \frac{\pi}{4}}$. And the maximum value is $\boxed{\|R_{\pi/4}\|_1 = \sqrt{2}}$.

Note that you can also derive this maximum value by calculus without using the above trig. identity, i.e., taking the derivative of $\cos\theta + \sin\theta$, setting it to 0, and finding the root, which is the extremal point.

Problem 5 (20 pts) Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an *orthonormal set* of an inner product space $\mathcal{V}$ with $\dim \mathcal{V} = n$.

(a) (5 pts) Prove that $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent if $k \le n$.

Answer: Let us form a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_k$,

$$\alpha_1 \mathbf{u}_1 + \cdots \alpha_k \mathbf{u}_k = \mathbf{0},$$

where α_j is a scalar, $1 \le j \le k$. Taking the inner product of the above with $\mathbf{u}_j$, we have

$$\begin{aligned}
\langle \mathbf{u}_j, \alpha_1 \mathbf{u}_1 + \cdots + \alpha_k \mathbf{u}_k \rangle &= \langle \mathbf{u}_j, \mathbf{0} \rangle \\
\langle \mathbf{u}_j, \alpha_1 \mathbf{u}_1 \rangle + \cdots + \langle \mathbf{u}_j, \alpha_k \mathbf{u}_k \rangle &= 0 \\
\alpha_1 \langle \mathbf{u}_j, \mathbf{u}_1 \rangle + \cdots + \alpha_n \langle \mathbf{u}_j, \mathbf{u}_k \rangle &= 0 \\
\alpha_j \langle \mathbf{u}_j, \mathbf{u}_j \rangle &= 0 \quad \text{due to the orthogonality} \\
\alpha_j &= 0 \quad \text{due to the normality,}
\end{aligned}$$

which is true for any $1 \le j \le k \le n$. Thus, $\{\mathbf{u}_j\}_1^k$ is a linearly independent set.

(b) (5 pts) For $k > n$, prove that $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly dependent.

Answer: Because the dimension of $\mathcal{V}$ is n, the maximum number of linearly independent vectors we can have is less than or equal to n. Therefore, if $k > n$, some of the vectors $\{\mathbf{u}_j\}_1^k$ must be linearly dependent.

(c) (5 pts) Suppose now $\mathcal{V} = \mathbb{R}^3$ and $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$, which are not an orthonormal set. Make these vectors orthonormal.

Answer: We apply the classical Gram-Schmidt procedure to these vectors. Let us denote these three vectors as $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be their orthonormal version. Then,

$$\mathbf{u}_1 = \frac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$\widetilde{\mathbf{u}}_2 = \boldsymbol{x}_2 - \langle \mathbf{u}_1, \boldsymbol{x}_2 \rangle \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Thus,

$$\mathbf{u}_2 = \frac{\widetilde{\mathbf{u}}_2}{\|\widetilde{\mathbf{u}}_2\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Now,

$$\begin{aligned} \widetilde{\mathbf{u}}_3 &= \boldsymbol{x}_3 - \langle \mathbf{u}_1, \boldsymbol{x}_3 \rangle \mathbf{u}_1 - \langle \mathbf{u}_2, \boldsymbol{x}_3 \rangle \mathbf{u}_2 \\ &= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{u}_3 \quad \text{already unit length.} \end{aligned}$$

Thus, we have the following orthonormal basis:

$$\boxed{\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}}.$$

(d) (5 pts) Expand a vector $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ with respect to the orthonormal vectors derived in Part (b).

Answer: We can expand this vector $\boldsymbol{x}$ easily as follows:

$$\begin{aligned} \boldsymbol{x} &= \langle \mathbf{u}_1, \boldsymbol{x} \rangle \, \mathbf{u}_1 + \langle \mathbf{u}_2, \boldsymbol{x} \rangle \, \mathbf{u}_2 + \langle \mathbf{u}_3, \boldsymbol{x} \rangle \, \mathbf{u}_3 \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mathbf{u}_1 + \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mathbf{u}_2 + \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mathbf{u}_3 \\ &= \boxed{\sqrt{2} \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2 + \mathbf{u}_3}. \end{aligned}$$