MAT 167: Homework Assignment #2 (due Monday, April 23)

First of all, do the following:

- Read Chapter 2.

Problem 1. Consider a matrix $A \in \mathbb{R}^{m\times n}$. Prove the column rank of A is the same as the row rank of A.

Problem 2. Prove that for a square matrix A, $\mathrm{null}(A) = \{\mathbf{0}\}$ implies A is invertible.

Problem 3. Find the minimum value of $\|\boldsymbol{x}\|_1$ subject to $\|\boldsymbol{x}\|_2 = 1$ in $\mathbb{R}^2$. Which $\boldsymbol{x}$ achieves such minimum?
[Hint: set $\boldsymbol{x} = [\cos\theta, \sin\theta]^T$, $0 \le \theta \le 2\pi$.]

Problem 4. Let $\|\cdot\|$ denote any norm on $\mathbb{R}^m$ and also the induced matrix norm on $\mathbb{R}^{m\times m}$. Let $\rho(A)$ be the *spectral radius* of A, i.e., $\rho(A) := \max\limits_{1\le i\le m} |\lambda_i(A)|$, where $\lambda_i(A)$ is the ith eigenvalue of A. Prove $\rho(A) \le \|A\|$.

Problem 5. Let $A = \boldsymbol{u}\boldsymbol{v}^T$ where $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{v} \in \mathbb{R}^n$. Prove $\|A\|_2 = \|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_2$.

Problem 6. **(a)** Define the following matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 1 & 3 \end{bmatrix},$$

in MATLAB. Then, compute the 2-norm by the `norm` function, and report the result in a long format (16 digits) via

```
>> format long
>> norm(A)
```

(b) Compute the 2-norm explicitly using the largest eigenvalue of A^TA using the `eig` function, i.e.,

```
>> sqrt(max(eig(A'*A)))
```

Then, compare the result with that of Part (a). What is the relative error between the norm computed in Part (a) and that in Part (b)?

(c) Compute the 1-norm, ∞-norm, and Frobenius norm of A by hand using the formulas derived in the class. Then, using the `norm` function, compare the MATLAB outputs with your hand-computed results. You should check how to use the `norm` function using the `help` utility:

```
>> help norm
```

(d) Let's load the MATLAB data file you used for HW1 again. It's located at
http://www.math.ucdavis.edu/~saito/courses/167.s12/hw01.mat .
Then, compute first the coefficient vector by

```
>> a = U'*x;
```

Now, compute $\|\boldsymbol{x}\|_p$ and $\|\boldsymbol{a}\|_p$, $p = 1, 2, \infty$, using the `norm` function, and report the results. Which value of p, you got $\|\boldsymbol{x}\|_p = \|\boldsymbol{a}\|_p$?

(e) Now, compute the matrix norms, $\|U\|_p$, $p = 1, 2, \infty$ as well as $\|U\|_F$ using the `norm` function, then report the results.