MAT 167: Homework Assignment #3 (due Monday, April 30)

First of all, do the following:

- Read Sec. 3.3, 3.6, and Chapter 4.

Problem 1. Using MATLAB, do the following procedure:

(a) Download the data file: http://www.math.ucdavis.edu/~saito/courses/167.s12/hw03.mat to your working directory, name it as hw03.mat. Then, load it into your MATLAB session. Check what variables (i.e., arrays) are defined in this data file by running:

```
>> whos
```

(b) Plot the data by:

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>> plot(x,y); grid;
```

(c) Create the Vandermonde matrix for polynomials of degree 1 (i.e., lines) by:

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>> A=[x.^0 x.^1];
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(d) Compute the least squares line over the given data by:

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>> sol = inv(A'*A)*A'*y;
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Then, overlay the least squares line over the current plot by:

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>> hold on; plot(x, sol(1)+sol(2)*x, '--');
```

Put title, axis labels by:

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>> title('Least squares line fit'); xlabel('x'); ylabel('y');
```

Finally, print out this plot and submit the hardcopy of the plot.

Problem 2. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \\ 1 & 1.0001 \end{bmatrix}; \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 0.0001 \\ 4.0001 \end{bmatrix}.$$

(a) What are the pseudoinverse $A^\dagger$ and the projector $P = AA^\dagger$ for this example? Give exact answers (not by MATLAB).

(b) Find the exact solutions $\boldsymbol{x}$ and $\boldsymbol{y} = A\boldsymbol{x}$ to the least squares problem $A\boldsymbol{x} \approx \boldsymbol{b}$ (not by MATLAB).

(c) The condition number of a rectangular shape matrix A is $\kappa(A) := \|A\| \, \|A^\dagger\|$. Compute the condition number of A in this example using MATLAB.

Problem 3. Let $A \in \mathbb{R}^{m \times m}$ be a *symmetric* matrix. As you already learned in MAT 22A or MAT 67, an *eigenvector* of A is a nonzero vector $\boldsymbol{x} \in \mathbb{R}^m$ such that $A\boldsymbol{x} = \lambda \boldsymbol{x}$ for some $\lambda \in \mathbb{C}$, the corresponding *eigenvalue*.

(a) Prove that all eigenvalues of A are real.

(b) Prove that if $\boldsymbol{x}$ and $\boldsymbol{y}$ are eigenvectors corresponding to distinct eigenvalues, then $\boldsymbol{x}$ and $\boldsymbol{y}$ are orthogonal.

Problem 4. If P is an orthogonal projector, then $I - 2P$ is an orthogonal matrix. Prove this algebraically first, and then give a geometric interpretation (i.e., using a simple figure).

Problem 5. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Answer the following by hand calculation.

(a) What is the orthogonal projector P_A onto $\operatorname{range}(A)$?

(b) What is the image under P_A of the vector $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$?

Problem 6. We want to prove the following statement: *If $A \in \mathbb{R}^{m \times n}$ is full rank, then $A^\top A$ is nonsingular (i.e., invertible).* Let's prove this step-by-step.

(a) Let $\boldsymbol{x}, \boldsymbol{z} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$, and consider the following:

$$\|A(\boldsymbol{x} + \alpha \boldsymbol{z}) - \boldsymbol{b}\|_2^2.$$

Write this quantity as a quadratic polynomial in terms of α.

(b) Suppose $\boldsymbol{x}$ solves the least squares problem, i.e., $\|A\boldsymbol{x} - \boldsymbol{b}\|_2^2$ is minimized while $\boldsymbol{x} + \alpha\boldsymbol{z}$ is not a solution to the lease squares problem. Then, using Part (a), deduce $A^\top(A\boldsymbol{x} - \boldsymbol{b}) = \boldsymbol{0}$, i.e., $\boldsymbol{x}$ is a solution to the normal equation $A^\top A\boldsymbol{x} = A^\top \boldsymbol{b}$.

(c) Suppose both $\boldsymbol{x}$ and $\boldsymbol{x} + \alpha\boldsymbol{z}$ are both solutions to the same least squares problem. Show that $\boldsymbol{z} \in \operatorname{null}(A)$.

(d) If A is full rank, then show that such $\boldsymbol{z}$ must be $\boldsymbol{0}$. This concludes the proof of the statement we wanted to prove, i.e., the normal equation has a unique solution, i.e., $A^\top A$ is nonsingular.