MAT 167: Homework Assignment #6 (due Monday, May 21)

First of all, do the following:

- Read Chapter 6.

Problem 1. Using MATLAB, do the following:

(a) Load the image called mandrill.mat, via:

```
>> load mandrill;
```

This loads a matrix X containing a face of mandrill, and a map containing the colormap of that image. If you cannot load this data in your MATLAB, then download this data from the following link: http://www.math.ucdavis.edu/~saito/courses/167.s12/mandrill.mat . Then, run the above load command again. Display this matrix on your screen by:

```
>> image(X); colormap(map)
```

Then, attach it in your HW sheets.

(b) Compute the SVD of this mandrill image and plot the distribution of its singular values on your screen (Note that the MATLAB `svd` function returns three matrices U, S, V for a given input matrix. So, the singular values are nicely plotted by:

```
>> stem(diag(S)); grid
```

Then print this figure and attach it in your HW sheets.

(c) Let σ_j, $\boldsymbol{u}_j$, $\boldsymbol{v}_j$ be the jth singular value, the jth left and right singular vectors of the mandrill image, respectively. In other words, they are `S(j,j)`, `U(:,j)`, `V(:,j)` of the SVD of X in MATLAB. Let us define the rank k approximation of the image X as

$$X_k := \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathsf{T}} + \cdots + \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k^{\mathsf{T}}.$$

Then, for $k = 1, 6, 11, 31$, compute X_k of the mandrill, and display the results. Fit these four images in one page by using subplot function in MATLAB (i.e., use `subplot(2,2,1)` to display the first image, `subplot(2,2,2)` to display the second image, etc.)

(d) For $k = 1, 6, 11, 31$, display the residuals, i.e., $X - X_k$, fit them in one page, print them, and attach that page in your HW sheets.

(e) For $k = 1, 6, 11, 31$, compute $\|X - X_k\|_2$ by the `norm` function of MATLAB. Then, compare the results with σ_{k+1}. More precisely, compute the relative error and report the results:

$$\frac{|\sigma_{k+1} - \|X - X_k\|_2\,|}{\sigma_{k+1}}.$$

Problem 2: Consider the matrix

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}.$$

(a) Determine an SVD of A by hand calculation. The SVD is not unique (module ± 1 factors), so find the one that has the minimal number of minus signs in U and V.

(b) List the singular values, left singular vectors, and right singular vectors of A. Draw a careful, labeled picture of the unit circle in $\mathbb{R}^2$ and its image under A, together with the singular vectors, with the coordinates of their vertices marked.

(c) What are the 1-, 2-, ∞-, and Frobenius norms of A?

(d) Find A^{-1} not directly (i.e., not using Cramer's rule), but via the SVD.

(e) Find the eigenvalues λ_1, λ_2 of A.

(f) Verify that $\det(A) = \lambda_1 \lambda_2$ and $|\det(A)| = \sigma_1 \sigma_2$.

(g) What is the area of the ellipsis onto which A maps the unit circle of $\mathbb{R}^2$?

Problem 3: Suppose $A \in \mathbb{R}^{m \times m}$ has an SVD $A = U \Sigma V^{\mathsf{T}}$. Find an *eigenvalue decomposition* of the $2m \times 2m$ symmetric matrix:

$$B = \begin{bmatrix} O & A^{\mathsf{T}} \\ A & O \end{bmatrix}.$$

Problem 4: Suppose A is a 202×202 matrix with $\|A\|_2 = 100$ and $\|A\|_F = 101$. Give the sharpest possible lower bound on the 2-norm condition number $\kappa(A)$.