MAT 167: Applied Linear Algebra
Lecture 21: Classification of Handwritten Digits

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May 17, 2017
Outline

1. The USPS Handwritten Digits Dataset
2. Simple Classification Algorithms
3. Classification Using SVD Bases
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The USPS Handwritten Digit Dataset

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This dataset is available from http://www.gaussianprocess.org/gpml/data/.

There are totally 9298 handwritten single digits between 0 and 9, each of which consists of $16 \times 16$ pixel image.

Half of 9298 digits are designated as training and the other half are as test: use 4649 digits for constructing a classification algorithm, and use the other 4649 digits to test the performance of that algorithm.

Pixel values are normalized to be in the range of $[-1, 1]$.

Each digit image is represented as a 1D vector of length 256 in the MATLAB file, so there are two matrices in the file called train_patterns and test_patterns each of which is of size $256 \times 4649$.

Note that if you want to see a digit of a particular column of these matrices using MATLAB like in the figure of the previous page, you need to reshape that column (a vector of length 256) to a small matrix of size $16 \times 16$ and then transpose it before rendering it using imagesc function.
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The labels are also known for both training and test sets. They are stored in `train_labels` and `test_labels` each of which is of size $10 \times 4649$.

Let $A \in \mathbb{R}^{10 \times 4649}$ be one of these label matrices. Then $A(:,j)$, i.e., $j$th column of $A$ describes the label of the $j$th digit in the following way. If that digit represents digit $i$ ($0 \leq i \leq 9$), then $A(i+1,j) = +1$ and $A(l,j) = -1$ for $l \neq i + 1$. 

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Notation

- Let $X = [x_1 \cdots x_n] \in \mathbb{R}^{d \times n}$ be the data matrix whose columns represent the digits in the \textit{training} dataset with $d = 256$, $n = 4649$.

- Let $Y = [y_1 \cdots y_m] \in \mathbb{R}^{d \times m}$ be the data matrix whose columns represent the digits in the \textit{test} dataset with $d = 256$, $m = 4649$. 
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2. Simple Classification Algorithms
3. Classification Using SVD Bases
The Simplest Classification Algorithm

The simplest classification algorithm is perhaps the following one:

1. Compute the mean (average) digits $m_i$, $i = 0, \ldots, 9$ using the training digits.
2. For each digit $y_j$, classify it as digit $k$ if $m_k$ is the closest mean.
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Figure: The mean digits (centroids) in the training dataset.
Note that there are many choices as the distance between each test and training digits, e.g., $\ell^p$-norm with $1 \leq p \leq \infty$, and cosine between them, etc., we decided to use the simplest one, i.e., the Euclidean, i.e., $\ell^2$ distance: $d(x_i, y_j) = \|x_i - y_j\|_2$.

The overall classification rate was 84.66%. More precisely:
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</table>
Figure: The worst test digits (the farthest from the means)
The Next Simplest Classification Algorithm

The next simplest one should be the so-called $k$-nearest neighbor ($k$-NN) classification algorithm as follows:

1. Select $k$ from small odd integers (i.e., 1, 3, 5, etc.)
2. For each test digit $y_j$, do:
   - Compute the distances from $y_j$ to all the training digits $\{x_i\}_{i=1,...,n}$.
   - Choose the $k$ nearest training digits from $y_j$.
   - Check the labels of these $k$ neighbors, and take a majority vote, which is assigned as a class label of $y_j$. 
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   - **Check** the labels of these *k* neighbors, and take a majority vote, which is assigned as a class label of *y*<sub>*j*</sub>.  

```bash
# pseudocode

1. Select k
2. For each y
   - Compute d(y, xi)
   - Choose k closest xi
   - Vote for class
```
**k-NN Classification Results**

- I tested with $k = 1, 3, 5$ using the MATLAB function `knnclassify` (in *Bioinformatics Toolbox*).
- The classification rates were considerably better than the previous simplest algorithm, i.e., 96.99%, 97.07%, 96.62%, respectively. More precisely, for $k = 3$: 
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  9 | 0 | 0 | 0 | 1 | 2 | 0 | 0 | 5 | 1 | 390|
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We use the first $k$ left singular vectors $\{u_1, \ldots, u_k\}$ of the SVD of the training digits. For each digit, we pool the training images corresponding to that digit, and compute the SVD. In other words, for each digit class, we compute the SVD.

Since we only need the first $k$ terms of the SVD, we only need to use `svds` function in MATLAB, which can specify $k$ as an input argument.

Let $X^{(j)}$ be a matrix of size $d \times n_j$ whose columns are training images corresponding to digit $j$ (hence $n_j$ is the number of training images corresponding to digit $j$), $j = 0, 1, \ldots, 9$.

Let the first $k$ terms of SVD of $X^{(j)}$ be $U_k^{(j)} \Sigma_k^{(j)} V_k^{(j)\top}$ where $U_k^{(j)} \in \mathbb{R}^{d \times k}$, $\Sigma_k^{(j)} \in \mathbb{R}^{k \times k}$, and $V_k^{(j)} \in \mathbb{R}^{n_j \times k}$.

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$U^{(j)}_k \Sigma^{(j)}_k V^{(j)^T}_k$ is the best rank $k$ approximation of $X^{(j)}$ in the sense of least squares.
The reasons why we use the first $k$ left singular vectors are:

- If $k$ is appropriately chosen, then $\text{range}(X^{(j)}) \approx \text{range}(U_k^{(j)})$. In fact, if $k = \min(d, n_j)$, then $\text{range}(X^{(j)}) = \text{range}(U_k^{(j)})$.

- The columns of $U_k^{(j)}$ are a part of ONB for $X^{(j)}$, which allows us to compute the $k$ expansion coefficients of each training image $x_i$ by simply multiplying (from left) $U_k^{(j)^T}$, i.e., $U_k^{(j)^T}x_i$ gives you such expansion coefficients.

- $U_k^{(j)}(U_k^{(j)^T}x_i)$ is the best $k$-term approximation in the least squares sense if $x_i$ belongs to digit $j$ class.

- Also, the SVD-based classification algorithm in the next page assume the following (if not, it won’t work well):
  - Each $X^{(j)}$ is well characterized and approximated by $U_k^{(j)}\Sigma_k^{(j)}V_k^{(j)^T}$ for the same value of $k$ for all 10 digits.
  - If you approximate $X^{(m)}$ using $U_k^{(j)}$ with $m \neq j$, the error will be large.
  - If an unlabeled test digit $y_l$ has the least $k$-term approximation error when using $U_k^{(j^*)}$, then it is likely that $y_l$ belongs to digit $j^*$. 
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- If you approximate $X^{(m)}$ using $U_k^{(j)}$ with $m \neq j$, the error will be large.

- If an unlabeled test digit $y_i$ has the least $k$-term approximation error when using $U_k^{(j^*)}$, then it is likely that $y_i$ belongs to digit $j^*$. 
The reasons why we use the first $k$ left singular vectors are:

- If $k$ is *appropriately* chosen, then $\text{range} \left( X^{(j)} \right) \approx \text{range} \left( U_k^{(j)} \right)$. In fact, if $k = \min(d, n_j)$, then $\text{range} \left( X^{(j)} \right) = \text{range} \left( U_k^{(j)} \right)$.

- The columns of $U_k^{(j)}$ are a part of ONB for $X^{(j)}$, which allows us to compute the $k$ expansion coefficients of each training image $x_i$ by simply multiplying (from left) $U_k^{(j)\top}$, i.e., $U_k^{(j)\top} x_i$ gives you such expansion coefficients.

- $U_k^{(j)} \left( U_k^{(j)\top} x_i \right)$ is the best $k$-term approximation in the least squares sense if $x_i$ belongs to digit $j$ class.

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  - If an unlabeled test digit $y_l$ has the least $k$-term approximation error when using $U_k^{(j*)}$, then it is likely that $y_l$ belongs to digit $j^*$. 
The reasons why we use the first $k$ left singular vectors are:

- If $k$ is appropriately chosen, then $\text{range}(X^{(j)}) \approx \text{range}(U^{(j)}_k)$. In fact, if $k = \min(d, n_j)$, then $\text{range}(X^{(j)}) = \text{range}(U^{(j)}_k)$.

- The columns of $U^{(j)}_k$ are a part of ONB for $X^{(j)}$, which allows us to compute the $k$ expansion coefficients of each training image $x_i$ by simply multiplying (from left) $U^{(j)}_k^T$, i.e., $U^{(j)}_k^T x_i$ gives you such expansion coefficients.

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An SVD Basis Classification Algorithm

- **Training:** Compute the best rank $k$ approximation of $X^{(j)}$, i.e.,
  $$U_k^{(j)} \Sigma_k^{(j)} V_k^{(j)T}.$$

- **Classification:** For a given test digit $y_l$, compute the 2-norm of residual errors,
  $$E_j(y_l) := \|y_l - U_k^{(j)} \left(U_k^{(j)T} y_l\right)\|_2, \quad j = 0, 1, \ldots, 9;$$
  If one of them, say, $E_{j^*}(y_l)$ is significantly smaller than all the others, then classify $y_l$ as digit $j^*$; otherwise give up.

**Note:** Mathematically, $y_l - U_k^{(j)} \left(U_k^{(j)T} y_l\right) = \left(I_d - U_k^{(j)} U_k^{(j)T}\right) y_l$, i.e., this is an orthogonal complement to the projection of $y_l$ onto range($U_k^{(j)}$). However, computationally, you should compute $U_k^{(j)T} y_l$ first, then multiply $U_k^{(j)}$. As I mentioned previously, if you first try to compute $U_k^{(j)} U_k^{(j)T}$, it would take a long time or even would be impossible to computing it if $d$ is large. This digit recognition problem has $d = 256$; so you can do either way.
An SVD Basis Classification Algorithm

- **Training:** Compute the best rank $k$ approximation of $X^{(j)}$, i.e.,
  \[ U_k^{(j)} \Sigma_k^{(j)} V_k^{(j)^T}. \]

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  \[ E_j(y_l) := \| y_l - U_k^{(j)} (U_k^{(j)^T} y_l) \|_2, j = 0, 1, \ldots, 9; \]
  If one of them, say, $E_{j^*}(y_l)$ is significantly smaller than all the others, then classify $y_l$ as digit $j^*$; otherwise give up.

**Note:** Mathematically,
\[ y_l - U_k^{(j)} (U_k^{(j)^T} y_l) = (I_d - U_k^{(j)} U_k^{(j)^T}) y_l, \]
 i.e., this is an orthogonal complement to the projection of $y_l$ onto range $U_k^{(j)}$. However, computationally, you should compute $U_k^{(j)^T} y_l$ first, then multiply $U_k^{(j)}$. As I mentioned previously, if you first try to compute $U_k^{(j)} U_k^{(j)^T}$, it would take a long time or even would be impossible to computing it if $d$ is large. This digit recognition problem has $d = 256$; so you can do either way.
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$U_{10}$ of Digits '0' and '1'
$U_{10}$ of Digits '2' and '3'
$U_{10}$ of Digits ’4’ and ’5’
$U_{10}$ of Digits '6' and '7'
$U_{10}$ of Digits '8' and '9'
SVD Classification Results with $k = 1:20$

Figure: $k = 17$ gave the best result: 96.62%
## Confusion Matrix for $k = 17$

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