Lecture 6: Interpolation & Approximation

* Interpolation vs Approximation

Consider a given set of pts \( \{(x_j, y_j)\}_{j=1:n} \subset \mathbb{R}^2 \).

- **Interpolation**: Find a fn \( y = f(x) \) s.t.
  \[ y_j = f(x_j), \quad j=1:n \]
  \[ \Rightarrow \text{Once you find } f(x), \text{ you can evaluate at any pt } x \notin \{x_j\} \]

- **Approximation (smoothing)**: Find a fn \( y = f(x) \) s.t.
  maximizing the fidelity to the data (i.e., minimizing the residual error) subj. to some smoothness constraint on \( f \). \( \Rightarrow \) This includes the least squares fit of a polynomial to the data, e.g.,
  \[ \sum_{j=1}^{n} (y_j - f(x_j))^2 \rightarrow \text{min!} \]
  subj. to \( f \in P_k \) (k th order polynomial) \( k \leq n-1 \)

We'll mainly focus on interpolation here.

Note: There is vast literature on both interp. & approx. Multi-dimensional cases are particularly important in geophysics, medicine, cartography, spatial statistics, and image processing ...

As you can imagine, mathematicians started analyzing these problems using algebraic poly's, trigonometric poly's, Chebyshev poly's, wavelets, rational func ...

- Runge
- Gibbs
Lagrange Polynomials / The Runge Phenomenon

Definition (Lagrange polynomial)

Given a set of points \( \{(x_j, y_j)\}_{j=0}^k \) \((k+1 \text{ pts})\),

let

\[
L_k(x) := \sum_{j=0}^k y_j \cdot l_j(x)
\]

where

\[
l_j(x) := \prod_{0 \leq m \leq k, m \neq j} \frac{x-x_m}{x_j-x_m}
\]

\[
= \frac{(x-x_0) \cdots (x-x_{j-1})(x-x_{j+1}) \cdots (x-x_k)}{(x_j-x_0) \cdots (x_j-x_{j-1})(x_j-x_{j+1}) \cdots (x_j-x_k)}
\]

Hence,

\[
l_j(x_m) = \delta_{jm} = \begin{cases} 1 & \text{if } m=j \\ 0 & \text{if } m \neq j \end{cases}
\]

\(\Rightarrow\) Kronecker's delta

This is a fundamental property for interpolation \( L_k(x_j) = y_j \)

Note:

- \(\deg(l_j) = k\).
- By defining \( \Phi(x) := \prod_{j=0}^k (x-x_j) \), we can write \( l_j(x) = \frac{\Phi(x)}{(x-x_j) \Phi'(x_j)} \).

The Runge Phenomenon (1901)

Carl Runge (1856 - 1927) reported the following observation: Consider \( f(x) = \frac{1}{1+25x^2} \) over \([-1, 1]\), and an interpolation poly. \( L_k \) at the equidistant nodes \( x_j = -1 + j \frac{2}{k}, j=0,1,\ldots, k \).

Then, the interpolating poly. \( L_k \) oscillates toward the edge of the interval \([-1, 1]\).

Moreover, \( \lim_{k \to \infty} \left( \max_{-1 \leq x \leq 1} |f(x) - L_k(x)| \right) = +\infty \).
Lagrange Polynomial Interpolation at 11 equispaced points to $1/(1+25x^2)$
Lagrange Polynomial Interpolation at 11 random points to $1/(1+25x^2)$
(Proof) First, I leave the following as an exercise:
\[
\frac{f^{(k+1)}(x)}{(k+1)!} = \Phi(x)
\]
Then
\[
\max_{x \in [-1,1]} |f(x) - L_k(x)| \leq \max_{x \in [-1,1]} \frac{1}{(k+1)!} \max_{j=0}^k |x - x_j|^{k+1}
\]
\[
\Rightarrow \text{Avoid global polynomial interpolation} \quad \text{(at least for equidistant pts)}
\]

- Hence, the idea of using piecewise algebraic polynomials was conceived.
- We'll defer interpolation using trigonometric poly. until we learn Fourier series.
- Here, our discussion is based on the paper by Micula (2002).

**Notation:** \( I := [a, b] \subset \mathbb{R} \)
\( \mathcal{P}_m := \{ \text{a set of poly's of deg.} \leq m \} \)
\( H^m(I) := \{ f: I \to \mathbb{R} \mid f^{(m-1)} \in AC(I) \& f^{(m)} \in L^2(I) \} \)
= \( L^2 \)-Sobolev space
\( W^{m,2}(I) \)
\( \Rightarrow f^{(m)} \) exists a.e. on \( I \)

**Def.** Let \( P := \{ a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b \} \)
be a partition of \( I \). Then \( s: I \to \mathbb{R} \)
is a polynomial spline of deg. \( m \) w.r.t. \( P \)
if \( s \in C^{m-1}(I) \) & \( s\big|_{[x_j, x_{j+1}]} \in P_m, j=0:n. \)

The interior pts \( \{x_1, \ldots, x_n\} \) are called knots.

**Natural Cubic Splines**

Suppose that the knots and the values at the knots \( \{y_1, \ldots, y_n\} \) are given.

Find a nice fcn \( y : I \rightarrow \mathbb{R} \) s.t. \( y(x_j) = y_j \)

\( j=1:n. \)

**Thm (Holladay 1957)**

Given \( \{(x_j, y_j)\}_{j=1:n}, n \geq 2, \)

\( \exists! \sigma \in X_n(I) := \{ f \in H^2(I) \mid f(x_j) = y_j, j=1:n \} \)

with \( \int_I |\sigma''(x)|^2 \, dx = \min_{f \in X_n(I)} \int_I |f''(x)|^2 \, dx \)

i.e., \( \sigma = \text{arg min}_{f \in X_n(I)} \int_I |f''(x)|^2 \, dx. \)

Furthermore,

\[
\begin{align*}
\{ & \sigma |_{[x_j, x_{j+1}]} \in P_3, j=1:n-1, \\
& \sigma |_{[a, x_1]}, \sigma |_{[x_n, b]} \in P_1 \}
\end{align*}
\]

“Natural” since outside of \([x_1, x_n]\), \( \sigma \) is linear.

Why would one choose to minimize \( \int_I |f''(x)|^2 \, dx \)?

(A1) The curvature of \( f \) is \( \frac{f''}{(1 + f'^2)^{3/2}}, \) which implies \( \sigma \approx \text{minimizes the total curvature if } \sigma \text{ is small!} \)

i.e., \( \sigma \) is smooth, no kinks.
(Note: Why the curvature of \( f \) is \( \frac{f''}{(1+f'^2)^{3/2}} \)?)

\[
\text{Ans: For a plane curve } s(t) = (x(t), y(t)), \text{ its curvature is } \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}, \text{ which is well-known. Then set } x(t) = t \text{ to get } \frac{\dot{y}}{(1+\dot{y}^2)^{3/2}}
\]

(A2) Nat. Cubic Spline \( \approx \) the solution of a physical problem of deflection of a bar of uniform, thin, elastic material without external force.

\[\Rightarrow \text{ min. elastic potential energy} \approx \min \int_1 \left| f''(x) \right|^2 \, dx \]

(A3) \( \phi \) can be interpreted as a curve closest to a straight line yet passing through those given pts.

Note: For each sub-interval, if we want

\[ I_j = \int_{x_{j-1}}^{x_j} f(x, y, y', y'') \, dx \to \min! \]

then the E-L eqn. becomes

\[ \frac{\partial f}{\partial y} - \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) + \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} = 0 \]

Deriving this is an excercise!

Now let \( f = y''^2 \). Then the E-L eqn. becomes \( y^{(4)} = 0 \) with an appropriate B.C.

\[ \Rightarrow \text{ Biharmonic eqn. (10)} \]
⇒ All sorts of generalization have been developed. See Micula and the references therein.

⇒ For computational algorithm, see spline fcn in MATLAB, and Algorithm 472 (ACM) by Herriot & Reinsch.

(Proof of the Hollanday Thm)
It suffices to prove for any \( f \in X_n(I) \),
\[
\int_I |f''(x)|^2 \, dx > \int_I |\sigma''(x)|^2 \, dx.
\]
Let \( \eta(x) := f(x) - \sigma(x) \), i.e., \( f(x) = \eta(x) + \sigma(x) \)
Since \( \sigma'' \equiv \text{const.} =: c_j \) in each \([x_j, x_{j+1}]\),
we have
\[
\int_I |f''(x)|^2 \, dx - \int_I |\sigma''(x)|^2 \, dx
= \int_I |\eta''(x)|^2 \, dx + 2 \int_I \eta''(x) \sigma''(x) \, dx
= \int_I |\eta''(x)|^2 \, dx + 2 \sum_{j=0}^{n} \int_{x_j}^{x_{j+1}} \eta''(x) \sigma''(x) \, dx
= \int_I |\eta''(x)|^2 \, dx + 2 \sum_{j=0}^{n} \left[ \eta'' \sigma'' \bigg|_{x_j}^{x_{j+1}} - \int_{x_j}^{x_{j+1}} \eta'' \sigma''' \, dx \right]
\tag{telescopic sum!}
= \int_I |\eta''(x)|^2 \, dx + 2 \left( \eta''(b) \sigma''(b) - \eta''(a) \sigma''(a) \right)
- 2 \sum c_j (\eta(x_{j+1}) - \eta(x_j))
= \int_I |\eta''(x)|^2 \, dx \text{ by setting } \sigma''(a) = \sigma''(b) = 0
> 0 \quad \text{hence, } \sigma: \text{linear in } [a,x_j], [x_n,b].