Lecture 8: Basics of PDEs II

The Uniqueness of the String Eqn.

Once a sol. is obtained via, e.g., sep. of var's, can we guarantee its uniqueness?
If yes, then we have peace of mind!

\[
\begin{align*}
\text{u}_{tt} &= \frac{c^2}{\text{p}} \text{u}_{xx} + \frac{1}{\text{p}} f(x,t), \quad x \in [x_1, x_2], \ t \geq t_0, \\
\text{B.C.} \quad u(x_1, t) &= u(x_2, t) = 0 \quad \text{(Dirichlet)} \\
\text{I.C.} \quad \{ u(x, t_0) = \phi(x), \quad \phi(x_i) = 0, \ i = 1, 2, \ \\
&\quad u_t(x, t_0) = \psi(x), \quad \psi(x_i) = 0, \}
\end{align*}
\]

Thm

Let \( D = [x_1, x_2] \times [t_0, \infty) \subset \mathbb{R}^2 \)

Let \( u \in C^2(D) \). If \( u(x, t) \) satisfies \((*)\), then it is the only fn in \( D \) with these properties!

(Proof) Let's assume \( \exists v \in C^2(D) \) s.t. \( u \neq v \).

Define \( \zeta(x, t) := u(x, t) - v(x, t) \).

Then \( \zeta \) is a sol. of

\[
\begin{align*}
\zeta_{tt} &= \frac{c^2}{\text{p}} \zeta_{xx} \quad \text{in } D \\
\zeta(x_i, t) &= 0, \quad i = 1, 2 \\
\zeta(x, t_0) &= 0 = \zeta_t(x, t_0)
\end{align*}
\]

\((**)\) easy to derive these since \((*)\) is linear!

To show: \( \zeta \) satisfying \((**) \equiv 0 \) in \( D \).

This is physically obvious since nothing really happens for \( \zeta \) with 0 B.C. & I.C.
Yet, we need to prove this mathematically.

Consider the total energy (\( = \text{kinetic + potential} \)) at any time \( t \geq t_0 \).
Then, “nothing happens” means that

\[ (***) \int_{x_1}^{x_2} \left[ \rho \frac{\partial^2 \xi}{\partial t^2} (x, t_1) + \tau \frac{\partial^2 \xi}{\partial x^2} (x, t_1) \right] dx = 0 \]

Total energy at \((x, t_1)\).

If we can show this, then we can say \(\xi(x, t) \equiv 0\) in \(D\).

To show (***) \(\), let's consider

\[ (**) \int_{t_0}^{t_1} \int_{x_1}^{x_2} \xi_t \left[ \rho \frac{\partial \xi}{\partial t} - \tau \frac{\partial \xi}{\partial x} \right] dx dt = 0 \]

\[ = \frac{1}{2} \frac{\partial}{\partial t} \left[ \rho \frac{\partial \xi}{\partial t} + \tau \frac{\partial \xi}{\partial x} \right] - \frac{\partial}{\partial x} \left( \tau \xi_t \xi_x \right) \]

\[ \left( \therefore \frac{1}{2} \left( 2 \rho \frac{\partial^2 \xi}{\partial t^2} + 2 \tau \frac{\partial \xi}{\partial t} \frac{\partial \xi}{\partial x} \right) \right. \]

\[ - \tau \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial x} - \tau \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial x} \]

\[ = \xi_t \left[ \rho \frac{\partial \xi}{\partial t} - \tau \frac{\partial \xi}{\partial x} \right] \]

So, \((***) = \int_{t_0}^{t_1} \int_{x_1}^{x_2} \left[ \frac{1}{2} \rho \frac{\partial^2 \xi}{\partial t^2} + \tau \frac{\partial \xi}{\partial x} \right] dx dt \]

Now, \(\int_{t_0}^{t_1} \int_{x_1}^{x_2} \left[ \right] dx dt = \int_{x_1}^{x_2} \int_{t_0}^{t_1} \left[ \right] dx dt \), so let's use

Green's Thm : \( \int_D \left( \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial t} \right) dx dt = f_1 |_{t_0} - f_1 \left|_{t_1} \right. \)

Set \(f_1 = -\tau \xi_t \xi_x\), \(f_2 = \frac{1}{2} \rho \xi_t^2 + \tau \xi_x^2\).

Then, \((***) = \int_{x_1}^{x_2} (-\tau \xi_t \xi_x) \left. dt - \frac{1}{2} \rho \xi_t^2 + \tau \xi_x^2 \right) dx \]

\( C = \int (-\tau \xi_t \xi_x) \left. dt + \int -\frac{1}{2} \rho \xi_t^2 + \tau \xi_x^2 \right) dx \)

on \(I (t = t_0, x_1 \leq x \leq x_2)\), \(\xi = 0 = \xi_t \) (I.C.)

\(II (t_0 \leq t \leq t_1, x = x_2)\) \(\} \xi = 0 \) (B.C.)

\(IV (t_0 \leq t \leq t_1, x = x_1)\) \(\} \xi = 0 \) (B.C.)

so, \(\xi_t = 0\).
Hence \( (\star) = -\frac{1}{2} \int_I \left( p \frac{\partial^2 \xi}{\partial x^2} + \tau \xi \frac{\partial \xi}{\partial x} \right) \, dx \)

\[ = \frac{1}{2} \int_x^2 \left( p \frac{\partial^2 \zeta(x, t_i)}{\partial x^2} + \tau \frac{\partial \zeta}{\partial x}(x, t_i) \right) \, dx \]

\[ = 0 \quad \forall t_i \geq t_0 \]

\[ \Leftrightarrow \zeta_t(x, t_i) = 0 = \zeta_x(x, t_i), \quad \forall (x, t_i) \in \mathbb{D} \]

\[ \Leftrightarrow \zeta(x, t) \equiv \text{a const. in } \mathbb{D} \]

By the B.C., \( \zeta(x_i, t) = 0, \ i = 1, 2 \).
This const. = 0, \ i.e., \( \zeta(x, t) \equiv 0 \) in \( \mathbb{D} \). \( \Box \)

\( \star \) Heat Conduction without Convection

We only consider heat conduction in a body whose material is homogeneous and isotropic.
(no inhomogeneity)
(e.g., no void, no composite material)

Remark: However, anisotropic diffusion has become an important & interesting topic in image processing (e.g., "Perona-Malik" model)

Let the quantity of heat \( \Delta Q \) penetrating an arbitrary chosen surface \( \Delta S \) within the medium in unit time, i.e., \( \Delta Q = \text{flux of heat through } \Delta S \).
Consider another surface element \( \Delta S \), \( \| \Delta S \) with the distance \( \Delta v \). Assume the temperature
\[ \{ u = u(x, y, z, t) = \text{const. on } \Delta S, \]
\[ u_1 = u(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) = \text{Const. on } \Delta S, \]
From experimental thermodynamics + kinetic theory of molecular motion, heat will flow from the surface of high temp. to that of low temp. in the direction \( \mathbf{n} \to \) those surfaces.

(The First Law of Thermodynamics)

Let \( \Delta u = u_1 - u \). Then we have

\[
\Delta Q \cdot \Delta \mathbf{v} \propto \Delta u \cdot \Delta S = k \Delta u \Delta S \quad (\ast)
\]

thermal conductivity

\( k = \) flux of heat when \( \Delta u = 1, \Delta S = 1, \) \& \( \Delta \mathbf{v} = 1 \).

Note that \( k \) in general depends on material and temp. We won't deal with \( k = k(u) \).

Instead, we will only deal with \( k = \text{Const} \).

From \((\ast)\), we have

\[
\Delta Q = k \frac{\Delta u}{\Delta v} \Delta S
\]

\[
\rightarrow k \frac{\partial u}{\partial v} \Delta S \quad \text{as} \quad \Delta v \rightarrow 0.
\]

normal deriv. of \( u \) on \( \Delta S \)

So,

\[
Q = k \iint_S \frac{\partial u}{\partial v} \, dS
\]

\[
= k \iint_S \mathbf{n} \cdot \nabla u \, dS
\]

the unit normal vector on \( S \)

Via the divergence thm for a vector field in \( \mathbb{R}^3 \)

\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_V \nabla \cdot \mathbf{F} \, dv
\]
and setting $F = \nabla u$, $V = \Delta u$, we have

\[
Q = k \iint_S \nabla u \cdot \nu \, dS = k \iiint_{\Delta u} \Delta u \, dV
\]

Remark: In LaTeX, $\Delta = \nabla \cdot \nabla : \text{Laplacian}$

\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2
\]

Apply the Mean Value Thm. now to get

\[
Q = k \iiint_{\Delta u} \nabla^2 u(\xi_i, \eta_j, \zeta_k, t) \, dV
\]

for $\exists (\xi_i, \eta_j, \zeta_k) \in \Delta V$ \hspace{1cm} (1)

1D version:

\[
f(c) = \frac{F(b) - F(a)}{b - a}, \exists c \in I = (a, b), \quad F' = f
\]

\[
\Leftrightarrow (b - a)f(c) = F(b) - F(a), \exists c \in I
\]

\[
\Leftrightarrow \text{vol}(I)f(c) = \int_I f(x) \, dx, \exists c \in I
\]

Now you can see how to get (1)

On the other hand, $u$ in $\Delta V$ is either increased or decreased by the accumulation or loss of quantity of heat in $\Delta V$ at the rate $\langle u_t \rangle$ (= the spatial average of $u_t$ over $\Delta V$). So, the flux of heat into/from $\Delta V$ is:

\[
\rho \langle u_t \rangle \Delta V \quad (2)
\]
σ = specific heat of the medium
= amount of heat necessary to raise the temp.
of a unit mass of medium by one unit temp.
in unit time.

Note in (2), \( ρ Δu = \text{mass of the medium} \).

Equating (1) = (2) and letting \( Δu \downarrow 0 \), we have

\[ \rho_0 σ u_t = k \nabla^2 u \]

or \[ u_t = \frac{k}{\rho_0} \nabla^2 u = \alpha^2 \Delta u, \quad \alpha^2 = \frac{k}{\rho_0} \]

Heat (or Diffusion) Eqn. !

If \( \exists \) heat source or sink at \((x, y, z)\) at time \( t \),
say, \( f(x, y, z, t) \), then the above PDE becomes

\[ u_t = \alpha^2 \Delta u + \frac{1}{\rho_0} f(x, y, z, t) \]

For the stationary temp. distrib., i.e., \( u_t \equiv 0 \),
we have \( \Delta u = -\frac{1}{k} f \) (Poisson's eqn.)

If \( \exists \) no \( f \), then this reduces to

\[ \Delta u = 0 \quad (\text{Laplace's eqn.}) \]