Derivatives, Integrals, & Uniform Convergence

Let \( f(\theta) \sim \sum c_n e^{i\theta} = \sum \hat{f}_n e^{i\theta} \).

And let's write
\[
f'(\theta) \sim \sum c'_n e^{i\theta} = \sum \hat{f}'_n e^{i\theta}
\]
\[
F(\theta) = \int_0^\theta f(\phi) d\phi \sim \sum C_n e^{i\theta} = \sum \hat{F}_n e^{i\theta}
\]

What are the relationship between \( c'_n, C_n \) and \( c_n \)?

Answer: \( c'_n = i n c_n, \quad C_n = \frac{c_n}{i n} \quad (n \neq 0) \)

You can see \( \frac{d}{d\theta} : \text{roughening} \)

\( \int_0^\theta : \text{smoothing} \)

More precisely,

Then let \( f: 2\pi - \text{periodic} \in C(\mathbb{R}) \cap PS(\mathbb{R}) \).

Then, \( c'_n = i n c_n \) \( \text{or} \hat{f}'_n = i n \hat{f}_n \).

In terms of \( a_n, b_n \), we have
\( a'_n = n b_n, \quad b'_n = -n a_n \quad \text{(Exercise!)} \)

(Proof) \[ C'_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(\theta) e^{-i\theta} d\theta \]

\[ \begin{aligned}
&= \frac{1}{2\pi} \left\{ \left[ f(\theta) e^{-i\theta} \right]_{-\pi}^{\pi} + i n \int_{-\pi}^{\pi} f(\theta) e^{i\theta} d\theta \right\} \\
&= \frac{1}{2\pi} \left\{ f(\pi) e^{-i\pi} - f(-\pi) e^{i\pi} \right\} + i n C_n \\
&= i n C_n \quad \text{(since } f: \text{ex. per. \& in } C(\mathbb{R}))
\]
As for the F.S. of an antiderivative of \( f \), note first that

an antiderivative of a periodic fcn
\( \neq \) a periodic fcn in general!

e.g., \( f(\theta) = 1 \) is a periodic fcn, but its antiderivative \( F(\theta) = \Theta \) is not periodic.

The key is \( c_0 = \hat{f}_0 = 0 \) or not.

Since \( \int_0^\infty c_n e^{in\phi} d\phi \) is 2\( \pi \)-periodic if \( n \neq 0 \),
we can see \( F(\theta) \) is 2\( \pi \)-periodic if \( c_0 = \hat{f}_0 = 0 \).

Thus, let \( f \in PC(\mathbb{R}), 2\pi\)-periodic, and
\[
F(\theta) = \int_0^\infty f(\phi) d\phi.
\]
If \( c_0 = \hat{f}_0 = 0 \), then
\[
F(\theta) = C_0 + \sum_{n \neq 0} \frac{a_n}{in} e^{in\theta} = \frac{A_0}{2} + \sum_{n=1}^\infty \left( \frac{a_n}{n} \sin n\theta - \frac{b_n}{n} \cos n\theta \right)
\]
where \( C_0 = \hat{F}_0 = \frac{A_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta) d\theta \).

(Proof) \( F(\theta + 2\pi) - F(\theta) = \int_\theta^{\theta+2\pi} f(\phi) d\phi = \int_{-\pi}^{\pi} f(\phi) d\phi \)
\[
= 2\pi C_0 = 0.
\]

So, \( F(\theta) \) is also 2\( \pi \)-periodic!

Moreover, \( F \in C(\mathbb{R}) \cap PS(\mathbb{R}) \) because \( f \in PC(\mathbb{R}) \).

Hence, the F.S. of \( F \) at \( \Theta = F(\Theta) \) \( \forall \Theta \in \mathbb{R} \).

By applying the previous Thm to \( F(\Theta) \), we get
\[
c_n = \hat{f}_n = i n C_n = i n \hat{F}_n \Rightarrow \hat{F}_n = \frac{\hat{f}_n}{i n}, n \neq 0
\]

These two thms suggest:
\[
\frac{d}{d\Theta} : \text{high freq. coeff. } \uparrow \quad ; \quad \int_0^\infty d\phi : \text{high freq. coeff. } \downarrow
\]
The previous corollary, i.e., $S_N[f](0) = \frac{1}{2}[f(0^-) + f(0^+)]$
for $f \in \text{PS}(\mathbb{R})$, $2\pi$-periodic, was about the pointwise convergence of F.S.
Now we want to have a theorem of absolute & uniform convergence!

**Def.** Suppose $\sum gn(x)$ converges to $g(x)$ on $x \in S$, $S$: some set. Then if $\sum |gn(x)|$ converges for $x \in S$ too, $\sum gn(x)$ is said to converge absolutely on $S$. If $\sup |g(x) - \sum_{n=0}^{\infty} gn(x)| \to 0 \ \forall x \in S$, then $\sum gn(x)$ is said to converge uniformly to $g(x)$ on $S$.

**Ex.** $g_n(x) = \frac{2(-1)^{n+1}}{n} \sin nx$, $S = [-\pi, \pi]$. $\Rightarrow \sum gn(x)$, i.e., the F.S. of $g(x) = x (2\pi \text{-per.})$, does not converge uniformly to $g(x)$ because the uniform limit of a continuous fun
must be continuous, but $g$ is discontinuous at $x = \pm \pi$.

To check the abs. & unif. conv. of a series, we can use the **Weierstrass M-test** (aka the **Comparison Test**): If $\exists M_n \geq 0$ is a reg. s.t. $|gn(x)| \leq M_n, \ \forall x \in S$ & $\sum M_n < \infty$, then $\sum gn(x)$ conv. abs. & unif. on $S$. 
So, in our case of F.S., because \( \sum_{n=-\infty}^{\infty} |c_n e^{-i\omega_0 n}| = 1 |c_n| \), if \( \sum_{n=-\infty}^{\infty} 1 |c_n| < \infty \), then the F.S. conv. abs. & unif. That is, 

**Thm (Sufficiency for abs. & unif. conv.)**

If \( f \in C(\mathbb{R}) \cap PS(\mathbb{R}), 2\pi \)-periodic, then the F.S. of \( f \) converges to \( f \) abs. & unif. on \( \mathbb{R} \).

(Proof) To show: \( \sum_{n=-\infty}^{\infty} |\hat{f}_n| < \infty \)

(Note that we already know \( \sum_{n=-\infty}^{\infty} |\hat{f}_n|^2 < \infty \) thanks to Bessel's inequality!)

Because of the cond. on \( f \), we have \( \hat{f}_n = i \text{Re} \hat{f}_n \Rightarrow |\hat{f}_n| = |\frac{\hat{f}_n}{n}| \) for \( n \neq 0 \).

Bessel's inequality applied to \( f \), gives us

\[
\sum_{n=-\infty}^{\infty} |\hat{f}_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta < \infty
\]

So, \[
\sum_{n=-\infty}^{\infty} |\hat{f}_n| = |\hat{f}_0| + \sum_{n \neq 0} |\frac{\hat{f}_n}{n}| \leq |\hat{f}_0| + \left( \sum_{n \neq 0} \frac{1}{n^2} \right)^{1/2} \left( \sum_{n \neq 0} |\hat{f}_n|^2 \right)^{1/2}
\]

\[
= |\hat{f}_0| + \frac{\pi}{\sqrt{3}} \left( \sum_{n \neq 0} |\hat{f}_n|^2 \right)^{1/2} < \infty
\]

**Remark:** This sufficient cond. is not necessarily the sharpest one. We won't prove the following sharper thm's:

**Thm (Bernstein, 1914)** \( f \in \text{Lip}_\alpha(\mathbb{R}) = C^\alpha(\mathbb{R}), \alpha > \frac{1}{2} \) \( \Rightarrow \) the F.S. of \( f \) conv. abs. & unif.

**Thm (Zygmund, 1928)** \( f \in \text{Lip}_\alpha(\mathbb{R}) \cap \text{BV}(\mathbb{R}), \alpha > 0 \) \( \Rightarrow \) the F.S. of \( f \) conv. abs. & unif.
Then (Smoothness class & Fourier coef)

Suppose \( f : 2\pi\text{-periodic}, \in C^{k-1}(\mathbb{R}), \) and 
\( f^{(k-1)} \in PS(\mathbb{R}), \) i.e., \( f^{(k)} \) exists except perhaps at finitely many pts in each bdd. interval.

Then, \( \sum_{n=0}^{\infty} |n^k \hat{f}_n|^2 < \infty. \)

In particular, \( n^k \hat{f}_n \to 0 \) as \( n \to \pm \infty \)

On the other hand, suppose the Fourier coef.'s satisfy \( |f_n| \leq C |n|^{-(k+\alpha)} \) \( \exists \alpha > 1 \) for \( n \in \mathbb{Z} \setminus \{0\} \). Then \( f \in C^k(\mathbb{R}) \).

(Proof) The first part: Apply the Deriv. Thm. \((\hat{f}_n = \hat{f}_{n-1})\)
\( k \) times to get \( f^{(k)} = (in)^k \hat{f}_n \).

Apply Bessel's ineq. to \( f^{(k)} \) to get
\[
\sum_{n=0}^{\infty} |n^k \hat{f}_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f^{(k)}(\theta)|^2 d\theta < \infty
\]

The second part:
\[
\sum_{n=0}^{\infty} |n^j \hat{f}_n| \leq C \sum_{n=0}^{\infty} |n|^{-(k-j+\alpha)} \leq 2C \sum_{n=0}^{\infty} n^{-\alpha} \text{ for } j \leq k
\]
\( \text{converges since } \alpha > 1. \)

\( \Rightarrow \sum (in)^j \hat{f}_n e^{in\theta} \text{ conv. abs. & unif. to } f^{(j)} \text{, } j \leq k. \)
\( \Rightarrow f^{(j)} \in C(\mathbb{R}), \text{ } j \leq k. \Rightarrow f \in C^k(\mathbb{R}). \)