Lecture 16: Fourier Series V

**Fourier Series on Intervals**

Suppose your fcn is defined on \([0, \frac{A}{2}]\) instead of \([-\frac{A}{2}, \frac{A}{2}]\).

\[ f_{\text{even}}(x) = \begin{cases} f(x) & x \in [0, \frac{A}{2}] \\ f(-x) & x \in [-\frac{A}{2}, 0] \end{cases} \]

\[ f_{\text{odd}}(x) = \begin{cases} f(x) & x \in (0, \frac{A}{2}] \\ 0 & x = 0 \\ -f(-x) & x \in [-\frac{A}{2}, 0) \end{cases} \]

\[ f \in C[0, \frac{A}{2}] \Rightarrow f_{\text{even}} \in C(\mathbb{R}, \text{A-periodic}) \]

\[ f \in C[0, \frac{A}{2}] \Rightarrow f_{\text{odd}} : \text{discontinuous} \]

Then their Fourier series expansions get simpler. But before computing them, let’s review the relationship between the Fourier coefficients \( \{ c_k \} \) w.r.t. the ONB \( \{ \frac{1}{\sqrt{A}} e^{2\pi i k x/A} \} \) and \( \{ a_k, b_k \} \) w.r.t. the ONB \( \{ \frac{1}{\sqrt{A}} \} \cup \{ \sqrt{\frac{2}{A}} \cos(\frac{2\pi k x}{A}) \} \cup \{ \sqrt{\frac{2}{A}} \sin(\frac{2\pi k x}{A}) \} \).

Let \( g(x) \) be an A-periodic \( L^2 \) fcn.

\[ g(x) \sim \frac{1}{\sqrt{A}} \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k x/A} \]

\[ c_k = \frac{1}{\sqrt{A}} \int_{-\frac{A}{2}}^{\frac{A}{2}} g(x) e^{-2\pi i k x/A} \, dx \]

Then \[ \frac{1}{\sqrt{A}} \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k x/A} \]

\[ = \frac{1}{\sqrt{A}} \left[ c_0 + \sum_{k=1}^{\infty} (c_k + c_{-k}) \cos \left( \frac{2\pi k x}{A} \right) + i (c_k - c_{-k}) \sin \left( \frac{2\pi k x}{A} \right) \right] \]
If we want, we can write \( a_0 \) instead of \( C_0 \).

**Remark:** In many books, the Fourier series is often defined in \([\pi, \pi]\) and written as

\[
\sum_{k=-\infty}^{\infty} c_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos kx + b_k \sin kx \right),
\]

where \( a_0 = 2C_0 \), \( a_k = C_k + C_{-k} \), \( b_k = i(C_k - C_{-k}) \), \( k \geq 1 \).

But \( \{ e^{ikx} \} \) is an orthogonal basis of \( L^2[-\pi, \pi] \), **but not normalized**. To make it an ONB, one needs the factor \( \frac{1}{\sqrt{2\pi}} \).

The same can be said for the orthogonal basis \( \{ 1 \} \cup \{ \cos kx \} \cup \{ \sin kx \} \).

The ONB is \( \{ \frac{1}{\sqrt{2\pi}} \} \cup \{ \frac{1}{\sqrt{\pi}} \cos kx \} \cup \{ \frac{1}{\sqrt{\pi}} \sin kx \} \), i.e., \( A = 2\pi \).

Compare this notation with mine in this lecture, which is the orthonormalized version:

\[
\sum_{k=-\infty}^{\infty} c_k \frac{1}{\sqrt{2\pi/kx/A}} e^{\frac{2\pi ikx}{A}} = \frac{1}{\sqrt{A}} a_0 + \frac{1}{\sqrt{A}} \sum_{k=1}^{\infty} \left( a_k \cos \left( \frac{2\pi kx}{A} \right) + b_k \sin \left( \frac{2\pi kx}{A} \right) \right)
\]

where \( a_0 = C_0 \), \( a_k = \frac{C_k + C_{-k}}{\sqrt{2}} \), \( b_k = \frac{C_k - C_{-k}}{\sqrt{2}} i \), \( k \geq 1 \).
Now, let’s go back to the Fourier series expansion of $f_{\text{even}}$ & $f_{\text{odd}}$.

$$c_k[f_{\text{even}}] = \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} f_{\text{even}}(x) e^{-2\pi i k x / A} \; dx$$

$$= \frac{2}{\sqrt{A}} \int_{0}^{A/2} f(x) \cos \left( \frac{2\pi k x}{A} \right) \; dx$$

$$= c_{-k}[f_{\text{even}}] \quad \text{thanks to the evenness of } \cos \theta.$$  

Recall the relationship:

$$a_0 = c_0, \quad a_k = \frac{c_k + c_{-k}}{\sqrt{2}}, \quad b_k = \frac{c_k - c_{-k}}{\sqrt{2}}.$$  

Hence, in this case of $f_{\text{even}}$, $a_0 = c_0[f_{\text{even}}]$, $a_k = \sqrt{2} c_k[f_{\text{even}}]$, $b_k \equiv 0$, $k \geq 1$.

In other words, $f_{\text{even}}$ can be written as the Fourier cosine series:

$$f_{\text{even}}(x) \sim \frac{1}{\sqrt{A}} \sum_{k=1}^{\infty} a_k \sqrt{\frac{2}{A}} \cos \left( \frac{2\pi k x}{A} \right)$$

where

$$a_0 = \frac{2}{\sqrt{A}} \int_{0}^{A/2} f(x) \; dx = c_0[f_{\text{even}}],$$

$$a_k = 2 \sqrt{\frac{2}{A}} \int_{0}^{A/2} f(x) \cos \left( \frac{2\pi k x}{A} \right) \; dx$$

$$= \sqrt{2} c_k[f_{\text{even}}].$$
Similarly, for \( f_{\text{odd}}(x) \),
\[
C_k[f_{\text{odd}}] = \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} f_{\text{odd}}(x) e^{-2\pi ikx/A} dx
\]
\[
= -\frac{2i}{\sqrt{A}} \int_0^{A/2} f(x) \sin\left(\frac{2k\pi x}{A}\right) dx
\]
\[
= -C_{-k}[f_{\text{odd}}] \quad \text{due to the oddness of } \sin \theta.
\]
\[
\Rightarrow a_k = \frac{C_k + C_{-k}}{\sqrt{2}} \equiv 0, \quad k \in \mathbb{N}.
\]
\[
a_0 = c_0 = \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} f_{\text{odd}}(x) dx = 0.
\]
\[
b_k = \frac{C_k - C_{-k}}{\sqrt{2}} i \equiv \sqrt{2} i C_k[f_{\text{odd}}]
\]

So,
\[
f_{\text{odd}}(x) \sim \sum_{k=1}^{\infty} b_k \frac{2}{\sqrt{A}} \sin\left(\frac{2\pi kx}{A}\right)
\]
\[
b_k = 2\frac{2}{\sqrt{A}} \int_{-A/2}^{A/2} f(x) \sin\left(\frac{2\pi kx}{A}\right) dx.
\]

So, if a function is given on \([0, A/2]\), say \( f \in C[0, A/2] \)
\(
\exists \text{ three ways to extend it to a periodic function:}
\)
\( O(1/n) \) (1) Brute force periodization with period \( A/2 \).
\( O(1/n^2) \) (2) Even extension followed by \( A \)-periodization.
\( O(1/n^3) \) (3) Odd extension followed by \( A \)-periodization.

(2) is the best among these 3 in terms of the decay of the Fourier coefficients.

However, \( \exists \) an even better way!
The Lanczos Method (1938)

Suppose \( f \in C^{2m}[0,1] \), but \( f(0) \neq f(1) \) no match at \( x=0,1 \). Also assume \( f^{(2m+1)} \in BV[0,1] \). \( m=1,2,\ldots \).

Lanczos's idea:

\[ f(x) = u(x) + v(x) \]

where \( u(x) \) is a polynomial of degree \( 2m-1 \).

\[
\begin{align*}
\int u^{(2k)}(0) &= f^{(2k)}(0), & k &= 0,1,\ldots, m-1, \\
u^{(2k)}(1) &= f^{(2k)}(1),
\end{align*}
\]

Then, consider the odd extension of \( v \).

\[ v \in C^{2m-1}(1\mathbb{R}), \quad v^{(k)}(0) = v^{(k)}(1) = 0 \]

\[ k = 0,1,2,\ldots, 2m-1. \]

And the Fourier sine coefficients of \( v(x) \) (with period 2) decay as

\[ b_k = O\left(|k|^{-2m-1}\right) \]

E.g., \( m=1 \) gives us \( b_k = O\left(\frac{1}{k^3}\right) \)

and \( u(x) \) is a straight line connecting \((0,f(0))\) & \((1,f(1))\).

Remarks:

1. \( f = u + v = (\text{an algebraic poly}) + (\text{a trig. poly}) \)
   can avoid both the Runge & Gibbs phenomena!

2. NS–J.F. Remy (2006) generalized this to \( \mathbb{R}^d \).