Lecture 19: Basics of $L^2$ Theory

Other Types of $L^2$ Spaces

1. **Weighted $L^2$ space**
   
   Let $w(x) \in (0, \infty)$ for a.e. $x \in [a, b]$. 
   
   $$L^2_w[a, b] := \{ f: [a, b] \to \mathbb{C} \mid \int_a^b |f(x)|^2 w(x) \, dx < \infty \}$$
   
   $$\langle f, g \rangle_w := \int_a^b f(x) \overline{g(x)} \, w(x) \, dx, \quad \|f\|_w := \sqrt{\langle f, f \rangle_w}$$
   
   $\Rightarrow$ Used frequently in Sturm–Liouville Theory Orthogonal Polynomials

2. **Higher-dimensions**

   Instead of the interval $[a, b]$, consider a domain $\Omega \subset \mathbb{R}^d$

   $$L^2(\Omega) := \{ f: \Omega \to \mathbb{C} \mid \int_{\Omega} |f(x)|^2 \, dx < \infty \}$$

   Thm. $L^2(\Omega)$ is complete.
   
   If $f \in L^2(\Omega)$, then $\exists \{ f_n \} \subset L^2(\Omega)$ s.t. $\|f_n - f\| \to 0$ as $n \to \infty$. Moreover, one can take $\{ f_n \} \subset C^0(\bar{\Omega}) \subset C^0(\Omega)$ have compact support in $\Omega$.

Hilbert Space

Def. A vector space $H$ is called a Hilbert space if

1. inner product is defined on elements in $H$
   
   (thus the norm is defined as $\|f\| = \sqrt{\langle f, f \rangle}$); and

2. $H$ is complete w.r.t. this norm.

Def. A vector space $B$ is called a Banach space if

1. a norm is defined on elements in $B$; and

2. $B$ is complete w.r.t. this norm.

Def. A set $M$ is called a metric space if

a distance (i.e., metric) is defined among elem's of $M$. 
\{ \text{Inner product spaces} \} \subset \{ \text{Named spaces} \} \subset \{ \text{Metric spaces} \} \\
\quad \cup \quad \cup \quad \cup \\
\{ \text{Hilbert spaces} \} \subset \{ \text{Banach spaces} \} \subset \{ \text{Complete metric spaces} \} \\
e.g. \quad L^2(\Omega) \quad \supset \quad L^p(\Omega), \quad 1 \leq p < \infty \quad \supset \quad IR \quad \text{with} \quad d(x,y) = |x-y|/1+|x-y| \\
\text{See highly informative math.stackexchange.com posting!}

Another important example of Hilbert space:
\[ l^2(\mathbb{N}) := \{ \mathbf{c} = (c_j)_{j=1}^{\infty}, \ c_j \in \mathbb{C} \mid \sum_{j=1}^{\infty} |c_j|^2 < \infty \} \]

For \( \mathbf{c}, \mathbf{d} \in l^2(\mathbb{N}) \), \( \langle \mathbf{c}, \mathbf{d} \rangle := \sum_{j=1}^{\infty} c_j \overline{d_j} \), \( \| \mathbf{c} \|_2 = \sqrt{\sum_{j=1}^{\infty} |c_j|^2} \).

Can show \( \{ \mathbf{c}_n \} \subset l^2(\mathbb{N}) : \text{Cauchy} \Rightarrow \mathbf{c}_n \rightarrow \mathbf{c} \in l^2(\mathbb{N}) \).

**Thm.** Any Hilbert space is isomorphic to \( l^2(\mathbb{N}) \).
(Hence, \( L^2(\Omega) \) is isomorphic to \( l^2(\mathbb{N}) \), which is referred to as the Riesz-Fischer Thm.)

**Def.** Two Hilbert spaces \( \mathcal{H} \) & \( \mathcal{H}' \) are said to be isomorphic to each other if \( \exists \) a bijection \( \Phi : \mathcal{H} \rightarrow \mathcal{H}' \) s.t.

1. \( \Phi(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha \Phi(\mathbf{x}) + \beta \Phi(\mathbf{y}), \ \forall \mathbf{x}, \mathbf{y} \in \mathcal{H}, \forall \alpha, \beta \in \mathbb{C} \).
2. \( \langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{H}} = \langle \Phi(\mathbf{x}), \Phi(\mathbf{y}) \rangle_{\mathcal{H}'} \).

(Proof) Suppose \( \{ \Phi_n \}_{n=1}^{\infty} \) be an ONB for \( \mathcal{H} \).

Then, we can consider a map \( \Phi : \mathcal{H} \rightarrow l^2(\mathbb{N}) \) by
\[ \Phi \mathbf{f} := \{ \langle \mathbf{f}, \Phi_n \rangle \}_{n=1}^{\infty}, \ \forall \mathbf{f} \in \mathcal{H}. \]

Thanks to Parseval's equality \( \| \mathbf{f} \|_N^2 = \sum_{n=1}^{\infty} |\langle \mathbf{f}, \Phi_n \rangle|^2 \), so clearly \( \Phi \mathbf{f} \in l^2(\mathbb{N}) \) and \( \| \mathbf{f} \|_N = \| \Phi \mathbf{f} \|_2 \) (isometry).

(2) is also satisfied via Parseval, \( \langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{H}} = \sum \langle \mathbf{f}, \Phi_n \rangle \langle \mathbf{g}, \Phi_n \rangle \).

So, \( \Phi \) is one-to-one from \( \mathcal{H} \) to \( l^2(\mathbb{N}) \). \( = \langle \Phi \mathbf{f}, \Phi \mathbf{g} \rangle_2 \) injection.
Remains to show: $\Phi$ is also "onto" (surjection).

Take any \( C = (c_1, c_2, \ldots) \in l^2(\mathbb{N}) \), i.e., \( \sum |c_j|^2 < \infty \).

Then consider the following sequence in \( \mathcal{F} \):

\[ f_1 = c_1 \phi_1, \quad f_2 = c_1 \phi_1 + c_2 \phi_2, \quad \ldots, \quad f_n = \sum_{j=1}^{c_n} c_j \phi_j, \quad \ldots \]

\[ \Rightarrow \text{For } m > n, \quad \| f_m - f_n \|_2^2 = \sum_{j=n+1}^{m} |c_j|^2 \to 0 \text{ as } m, n \to \infty. \]

So, \( \{f_n\} \): Cauchy in \( \mathcal{F}. \quad \{\phi_j\} \): ONB

That implies that \( f_n \to \exists! f \in \mathcal{F} \) thanks to its completeness.

For this \( f \), we clearly have \( \Phi f = C \), and \( C \in l^2(\mathbb{N}) \) was arbitrary. Hence we are done!

\[ \star \text{ The Dominated Convergence Thm} \]

Let \( \Omega \subset \mathbb{R}^d \), \( d \in \mathbb{N} \), be a domain, and let \( \{g_n\}, g, \phi \) be funcns on \( \Omega \) s.t.

\[ \begin{cases} (a) \, \phi(x) \geq 0 \text{ and } \int_{\Omega} \phi(x) \, dx < \infty; \\ (b) \, |g_n(x)| \leq \phi(x), \quad \forall n \in \mathbb{N}, \forall x \in \Omega; \quad \text{and} \\ (c) \, g_n(x) \to g(x), \quad \text{a.e. } x \in \Omega. \end{cases} \]

Then, \( \int_{\Omega} g_n(x) \, dx \to \int_{\Omega} g(x) \, dx \).

\[ \text{Ex. 1: } d = 1, \, \Omega = \mathbb{R}. \quad g_n(x) = e^{ixn} \phi(x), \quad \phi \geq 0, \int \phi < \infty. \]

Then, \( |g_n(x)| \leq |e^{ixn} \phi(x)| = |\phi(x)| = \phi(x) \).

Also, we see \( g_n(x) \to \phi(x) \), so by the D.C. Thm, \( \lim_{n \to \infty} \int_{\Omega} g_n(x) \, dx = \int_{\Omega} \lim_{n \to \infty} g_n(x) \, dx = \int_{\Omega} \phi(x) \, dx < \infty. \]

\[ \text{Ex. 2: } \text{Consider the funcn in the figure.} \]

\( g_n(x) \to 0 \) a.e., but \( \int_{\Omega} g_n(x) \, dx = \frac{1}{2} \cdot n \cdot \frac{2}{n} = 1. \)

So, \( \lim_{n \to \infty} \int_{\Omega} g_n(x) \, dx = 1. \) On the other hand,
there is no \( \phi \) satisfying (a), (b). Moreover, \( \int_\Omega \lim g_n(x) dx = 0 \).
The D.C. Thm doesn't hold. The idea of \( \mathcal{S} \text{Fen} \) & the theory of distribution is necessary to deal with such an example!

Now, recall the following relationship:

\[ \text{L}^2\text{-Norm conv.} \quad \xrightarrow{\text{1}} \quad \text{Pointwise conv.} \quad \xrightarrow{\text{2}} \quad \text{Uniform conv.} \]

With a bit more assumption, however, we can show Pointwise conv. \( \Rightarrow \) L^2\text{-norm conv.} as follows:

Then let \( \{ f_n \} \subset L^2(\Omega) \), \( f_n \to f \) pointwise.

If \( \exists y \in L^2(\Omega) \) s.t. \( |f_n(x)| \leq |y(x)| \) a.e. \( x \in \Omega \), then \( f_n \to f \) in norm.

(Proof) \( |f(x)| = \lim_{n \to \infty} |f_n(x)| \leq |y(x)| \) a.e. \( x \in \Omega \).

So, \( |f_n(x) - f(x)|^2 \leq (|f_n(x)| + |f(x)|)^2 \leq 2 |y(x)|^2 \).

Now apply the D.C. Thm with \( g_n(x) = |f_n(x) - f(x)|^2 \), \( g(x) = 0 \), and \( \phi(x) = 124(x)|^2 \) to get:

\[ \lim \int_\Omega g_n(x) dx = \int_\Omega \lim g_n(x) dx = \int_\Omega g(x) dx = 0 \]

\[ = ||f_n - f||^2_{L^2(\Omega)} \]

So, \( ||f_n - f||^2 \to 0 \). \( \blacksquare \)
**Best Approximation in $L^2$**

- If $\{\phi_n\}$ is an ONB of $L^2(\Omega)$, then $f = \sum \left< f, \phi_n \right> \phi_n \quad \forall f \in L^2(\Omega)$.

- If $\{\phi_n\}$ is an ON set, but not complete in $L^2(\Omega)$, then $\forall f \in L^2(\Omega)$, $\exists$ residual error $\hat{f} = f - \sum \left< f, \phi_n \right> \phi_n$ and $\sum \left< f, \phi_n \right> \phi_n = \sum \hat{f} \in L^2(\Omega)$.

The last lemma in Lecture 18.

It turns out that $\hat{f}$ is the best linear approx. of $f$ in $L^2(\Omega)$ in the following sense:

**Thm** $\| f - \sum \left< f, \phi_n \right> \phi_n \| \leq \| f - \sum \alpha_n \phi_n \|$ for arbitrary choice of $\{\alpha_n\}$ with $\sum |\alpha_n|^2 < \infty$.

- Holds iff $\alpha_n = \left< f, \phi_n \right>$, $\forall n$.

(Proof) This is really the least squares approx.!

$$f - \sum \alpha_n \phi_n = f - \sum \left< f, \phi_n \right> \phi_n + \sum \left( \left< f, \phi_n \right> - \alpha_n \right) \phi_n$$

$\perp$ to $\phi_j \in \{\phi_n\}$ a linear comb. of $\{\phi_n\}$

$\therefore \left< f, \phi_j \right> - \sum \left< f, \phi_n \right> \phi_n = \delta_{nj}$

So, the Pythagorean Thm applies:

$$\| f - \sum \alpha_n \phi_n \|^2 = \| f - \sum \left< f, \phi_n \right> \phi_n \|^2 + \sum |\left< f, \phi_n \right> - \alpha_n|^2$$

$\geq \| f - \sum \left< f, \phi_n \right> \phi_n \|^2$ $\geq 0$

and clearly $\implies$ holds iff $\alpha_n = \left< f, \phi_n \right>$. $\blacksquare$

**Cor.** $\{\phi_n\}^\infty$: an ONB for $L^2(\Omega)$. Then for any $f \in L^2(\Omega)$, the $N$th partial sum $\sum \left< f, \phi_n \right> \phi_n$ is the best linear approx. in $L^2$-norm to $f$ among all linear combinations of $\{\phi_1, \ldots, \phi_N\}$.

Note that $\{\phi_1, \ldots, \phi_N\}$ are selected independently from $f$. 
If we choose \( N \) basis func dependent on \( f \), i.e.,
\[
\{ \phi_{x_1}, \ldots, \phi_{x_N} \} \subset \{ \phi_n \},
\]
where \( x_1, \ldots, x_N \) depend on \( f \),
then \( \sum_{j=1}^{N} \langle f, \phi_{x_j} \rangle \phi_{x_j} \) is better than \( \sum_{j=1}^{N} \langle f, \phi_j \rangle \phi_j \) in general.

Example (Obvious) Say \( N = 100 \), \( f(x) = \phi_{101}(x) \).
Then \( f - \sum_{j=1}^{100} \langle f, \phi_j \rangle \phi_j = \phi_{101} \neq 0 \).

But clearly \( x_1 = 101 \), and \( f = \phi_{101} \) is just a one term!
This way of approximation is called \textit{nonlinear approx.}
In practice, \( \{ x_1, \ldots, x_N \} \) are chosen so that \( |\langle f, \phi_{x_j} \rangle| \)
are the largest \( N \) expansion coeff's of \( f \) w.r.t. \( \{ \phi_n \} \).