MAT 207B Methods of Applied Mathematics
Homework 4: due Friday, 02/09/18

Problem 1: Do Problems 28 and 32 on page 69.

Problem 2: Do Problem 38 on page 74.

Problem 3: Do Problem 39 on page 79.

Problem 4: Consider the problem

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + u = 0, \quad x \in [0, \ell],$$

with the homogeneous Dirichlet boundary condition: $u(0) = u(\ell) = 0$. Clearly, $u(x) \equiv 0$ is a solution. Is this solution *unique or not*? Does the answer depend on ℓ?

Problem 5: Let $\Omega \in \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Show that the following two *Neumann* problems are roughly equivalent (i.e., if you can find a solution to one of them, then you can also solve the other).

$$\begin{cases} \Delta v = f & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases} \tag{2}$$

You can assume that there exists $\widetilde{g} \in C^2\left(\overline{\Omega}\right)$, an extension of g, such that $\dfrac{\partial \widetilde{g}}{\partial \nu} = g$ on $\partial\Omega$.

[Hint: Tailor what I did in class for the Dirichlet problems.]

Problem 6: Prove that the surface area and the volume of the unit ball $B_n(0,1) \in \mathbb{R}^n$ is

$$|\partial B_n(0,1)| =: \omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \quad |B_n(0,1)| = \frac{\omega_n}{n},$$

respectively.

Problem 7: Suppose $f(x) = \sum_{k=1}^{n} b_k \sin \pi k x$, $x \in [0,1]$, for some $b_k \in \mathbb{R}$, $k = 1, \ldots, n$. Solve the following Dirichlet problem in the unit square $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u(x,0) = 0 & x \in [0,1] \\ u(0,y) = u(1,y) = 0 & y \in [0,1] \\ u(x,1) = f(x) & x \in [0,1]. \end{cases}$$