MAT 207B Methods of Applied Mathematics
Homework 8: due Friday, 03/09/18

Problem 1: Given the differential equation

$$f'' + P(x)f' + Q(x)f = g(x), \quad a \le x \le b,$$

find a function p on $[a,b]$ such that the given equation is equivalent to

$$(pf')' + qf = pg$$

for a suitable function q on $[a,b]$. Assume P is continuous.

Problem 2: Show that, if φ is an eigenfunction corresponding to an eigenvalue λ of the Sturm-Liouville system:

$$\begin{cases} (xf')' + \lambda x f = 0, & 0 < x \le R; \\ f : \text{bounded on } (0,R], & f(R) = 0, \end{cases}$$

then

$$\lambda \int_0^R x|\varphi(x)|^2 \, \mathrm{d}x = \int_0^R x|\varphi'(x)|^2 \, \mathrm{d}x.$$

Then, deduce that all eigenvalues of the system are *positive*.

Problem 3: Prove that the *singular* Sturm-Liouville system

$$\begin{cases} (x^2 f')' + \lambda f = 0, & 0 < x \le 1; \\ f : \text{bounded on } (0,1], & f(1) = 0, \end{cases}$$

has *no* eigenvalues.

[Hint: Find $\alpha \in \mathbb{C}$ such that $f(x) = x^\alpha$ is a solution of the differential equation, and obtain the general solution by putting $f = x^\alpha u$.]

Problem 4: Let $k > 0$ be a constant and let $f \in C[0,\infty)$. Show that

$$u(t) = \frac{1}{k} \int_0^t \sin k(t-s) f(s) \, \mathrm{d}s$$

is a solution of the following *initial value problem* (IVP):

$$\begin{cases} u'' + k^2 u & = f(t), \quad t \ge 0; \\ u(0) = u'(0) & = 0. \end{cases}$$

Problem 5: Let ω be positive, but not an integer multiple of π. Find the *Green's function* for the boundary value problem:

$$\begin{cases} f'' + \omega^2 f = g \\ f'(0) = 0 = f'(1). \end{cases}$$

What happens if we try this with $\omega = 0$?

Problem 6: Find the *Green's function* for the inhomogeneous Sturm-Liouville problem:

$$\begin{cases} f'' = g \\ f(0) = 0,\ f(1) + f'(1) = 0. \end{cases}$$