

MAT 207B Methods of Applied Mathematics

Homework 9 (The Last One!): due Friday, 03/16/18

Problem 1: The integral operator \mathcal{K} on $L^2(0, \infty)$ is defined by

$$\mathcal{K}f(x) := \int_0^\infty k(x+\tau)f(\tau) d\tau,$$

where $k \in C(0, \infty)$ is a complex-valued function such that

$$M := \int_0^\infty \tau |k(\tau)|^2 d\tau < \infty.$$

Show that \mathcal{K} is Hilbert-Schmidt with Hilbert-Schmidt norm \sqrt{M} . Note that the *Hilbert-Schmidt norm* of a general integral operator $\mathcal{K} : L^2(a, b) \rightarrow L^2(c, d)$ with its kernel $k(\cdot, \cdot) \in L^2((c, d) \times (a, b))$ is defined as

$$\|\mathcal{K}\|_{\text{HS}} := \sqrt{\int_c^d \int_a^b |k(x, \tau)|^2 dx d\tau}.$$

Problem 2: By applying the Sturm-Liouville Theorem to the system

$$\begin{cases} f'' + \lambda f = 0, & 0 \leq x \leq \pi; \\ f(0) = 0 = f'(\pi), \end{cases}$$

show that for any $g \in L^2(0, \pi)$,

$$g(x) = \sum_{j=1}^{\infty} c_j \sin\left(j - \frac{1}{2}\right)x$$

in the norm of $L^2(0, \pi)$ where

$$c_j = \frac{2}{\pi} \int_0^\pi g(x) \sin\left(j - \frac{1}{2}\right)x dx, \quad j \in \mathbb{N}.$$

Problem 3: Assume the following fact: $\{\phi_k(x) := J_0(z_k x/b)\}_{k \in \mathbb{N}}$ forms an orthogonal basis for $L^2_w(0, b)$ with $w(x) = x$ and z_k as the k th zero of $J_0(x)$, and $\|\phi_k\|_w^2 = \frac{b^2}{2} J_1(z_k)^2$ where J_1 is the Bessel function of the first kind of order 1. Then, expand the characteristic function $\chi_{[0, b/2]}(x)$ as a *Fourier-Bessel series* $\sum_{k=1}^{\infty} c_k J_0(z_k x/b)$; that is, compute the coefficients c_k , $k \in \mathbb{N}$.

[Hint: You may want to use the following famous formula:

$$\frac{d}{dx} [x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x), \quad \forall x \in \mathbb{R}; \forall \nu \in \mathbb{R}.$$

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Problem 4: Make the solution of the hanging chain problem given in the last theorem in Lecture 25 *explicit* by using the expressions for λ_j and φ_j in terms of J_0 , i.e., the Bessel function of the first kind of order 0 and its zeros z_j .