MAT 207B Methods of Applied Mathematics
Homework 1: due Friday, 01/17/25

Problem 1: Show that a moving particle with mass m obeys Newton's equation of motion: $m\ddot{\boldsymbol{r}} = -\nabla V(\boldsymbol{r})$, then it is a conservative field, i.e., $E = T + V$ is a constant independent of time t.

[Hint: Take the time derivative of E.]

Problem 2: Let $\boldsymbol{r}(t) \in \mathbb{R}^3$ be the position of a particle of mass m at time t under the conservative potential field $V(\boldsymbol{r})$ such that the usual Newton's equation of motion

$$m\ddot{\boldsymbol{r}} = -\nabla V \tag{1}$$

holds. Consider the polar coordinate system (r, θ, φ) as follows:

$$\begin{cases} x = r\sin\theta\cos\varphi; \\ y = r\sin\theta\sin\varphi; \quad 0 \le \theta \le \pi, 0 \le \varphi < 2\pi; \\ z = r\cos\theta. \end{cases}$$

Convert (1) from the (x, y, z)-coordinate system to the polar coordinate system by computing the first and second time derivatives of the variables (r, θ, φ) and write out three equations in those variables. Note that the computations may be tedious.

[Hint: You may need to use the following relationships:

$$\begin{cases} r^2 = x^2 + y^2 + z^2; \\ \tan\varphi = y/x; \\ \tan^2\theta = (x^2 + y^2)/z^2. \end{cases}$$

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Problem 3: Consider the motion of the same particle as Problem 1.

(a) Represent the Lagrangian $L = T - V = \frac{1}{2}m|\dot{\boldsymbol{r}}|^2 - V$ using the polar coordinate variables (r, θ, φ).

(b) Derive the Euler-Lagrange Equation for each variable in (r, θ, φ), and confirm that these agree with the equations of motion you derived in Problem 2.

Problem 4: **(a)** Is Newton's gravitational field, which is given by $F = \frac{\boldsymbol{r}}{r^3}$, where $\boldsymbol{r} = x\mathbf{i} + \mathbf{yj} + \mathbf{zk}$ and $r = |\boldsymbol{r}|$, conservative?

(b) If (a) permits a positive answer, find the potential function which characterizes Newton's gravitational field.

Problem 5: Give a geometric interpretation of the variational problem

$$\int_0^1 \sqrt{1+y'^2}\,\mathrm{d}x \to \text{minimum}$$

with the boundary conditions $y(0)=0,\ y(1)=1$.

Problem 6: Find the differential equation of a path down which a particle will fall from one given point to another in the shortest possible time. (*Hint:* It follows from the equation of motion of free-falling bodies that $v=\sqrt{2gy}$, where v is the velocity and y the height. Since $\mathrm{d}s/\mathrm{d}t=v$, we have

$$t=\int_{x_1}^{x_2} \frac{\mathrm{d}s}{\sqrt{2gy}}$$

where x_1 and x_2 are the x-coordinates of the beginning and the end point of motion.)
Remark: The solution of this problem (a *cycloid*), which was first proposed "to the mathematicians of the world to give their consideration" by *John Bernoulli* in 1696, is called *Brachistochron* ($\beta\rho\alpha\chi\iota\sigma\tau o\zeta$ = shortest, $\chi\rho\acute{o}\nu o\zeta$ = time).

Problem 7: Find the Euler-Lagrance equation of the variational problem

$$\int_{x_1}^{x_2} f(x,y,y',y'')\,\mathrm{d}x \to \text{minimum}$$

with the boundary conditions

$$y(x_1)=y_1,\quad y(x_2)=y_2,$$
$$y'(x_1)=y_1',\quad y'(x_2)=y_2'.$$

Problem 8: The potential energy (deformation energy) of an elastic (laterally movable) rod is given by

$$V=\int_0^l \kappa^2\,\mathrm{d}x$$

where κ is the curvature of the rod, $y=y(x)$, and the interval $0\le x\le l$ is the projection of the rod onto the x-axis.
Find the Euler-Lagrance equation for the variational problem

$$V \to \text{minimum}$$

and state suitable boundary conditions. (See Problem 7).

Problem 9: Find the Euler-Lagrange equation for the following variational problem:

$$\iint_R \sqrt{1+(\partial u/\partial x)^2+(\partial u/\partial y)^2}\,\mathrm{d}x\,\mathrm{d}y \to \text{minimum}$$

with the boundary condition $u(x,y)=1$ on $x^2+y^2=1$.