MAT 207B Methods of Applied Mathematics
Homework 2: due Friday, 01/24/25

Problem 1: A flexible rope of length $2a$ made of uniform material is hanged between the two points $(x, y) = (0, 0)$ and $(x, y) = (2b, 0)$ with $b < a$. The potential energy of this rope is:

$$V = \int_0^{2b} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, y\, \mathrm{d}x$$

Find a curve of $y = y(x)$ of this rope that minimizes this potential energy (which is called a *catenary*). Note that we have the following relationship:

$$\int_0^{2b} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x = 2a.$$

Problem 2: Show that the problem of of finding a curve which shall encompass a given area, while having the smallest possible length, is equivalent to the problem of Dido. (*Hint:* Compare the Euler-Lagrange equations of the two problems).

Problem 3: Let p, q, r be given functions of x. Find the Euler-Lagrange equation of the isoperimetric problem

$$\int_{x_1}^{x_2} (p y'^2 - q y^2)\, \mathrm{d}x \to \text{minimum},$$

subject to

$$\int_{x_1}^{x_2} r y^2\, \mathrm{d}x = 1, \quad y(x_1) = y_1, \quad y(x_2) = y_2.$$

Problem 4: Find the Euler-Lagrange equation of the two dimensional isoperimetric problem

$$\iint_R f\left(x, y, z(x,y), \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \mathrm{d}x\, \mathrm{d}y \to \text{minimum}$$
$$\iint_R g\left(x, y, z(x,y), \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \mathrm{d}x\, \mathrm{d}y = L,$$

where $z = z(x, y)$ satisfies the boundary condition

$$z(x, y) = z_0(x, y) \text{ on } C, \text{ where } C \text{ is the boundary of } R.$$

Problem 5: Consider a river with parallel banks, whose velocity $\boldsymbol{v}(x, y)$ at any point is given by

$$\boldsymbol{v} = x(x-1)\boldsymbol{j},$$

and a dog whose capable of swimming with a constant speed of $c = 1$. Formulate a variational problem to find the path $y(t)$ the dog should follow to cross the river, from $x_1 = 0$ to $x_2 = 1$, in the shortest amount of time.

Problem 6: What are the Euler-Lagrange equation and the natural boundary conditions for problem 5?

Problem 7: Find the partial differential equation (with an appropriate boundary condition) whose solution $u = u(x, y)$ minimizes the following functional

$$I = \iint_D \left(u_{xx} + u_{yy}\right)^2 \,\mathrm{d}x\,\mathrm{d}y = \iint_D \|\Delta u\|^2 \,\mathrm{d}x\,\mathrm{d}y$$

subject to the natural boundary condition where $D \in \mathbb{R}^2$ is a simply-connected domain. Note that the solution of such PDE leads to the natural spline functions in 2D corresponding to the cubic spline functions in 1D.