MAT 207B Methods of Applied Mathematics
Homework 4: due Friday, 02/07/25

Problem 1: Consider a half space, bounded by a plane and assume that heat flows only in the direction perpendicular to this plane. Let the boundary plane coincide with the y,z-plane and state the heat equation. Also state reasonable boundary and initial conditions.

Problem 2: State the boundary and initial value problem for the temperature distribution in the interior of an infinite cylinder $x^2+y^2=1$, if the temperature on the walls of the cylinder is given by

$$\Phi(x,y,z,t)=\frac{t+1}{t+z^2+1},$$

and the temperature in the interior at time $t=0$ is everywhere $1/(z^2+1)$.

Problem 3: Prove the following theorem: *If $u(x,y)$ is a solution of $\Delta u=0$ in a bounded two dimensional region R which is encompassed by the rectifiable curve C, is continuous and has continuous partial derivatives in R and on C and satisfies on C the conditions $u=\Phi$ or $\partial u/\partial n=\Psi$, then it is the only solution except for an additive constance in the case of the latter boundary value problem.*

Problem 4: Consider the problem

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+u=0,\qquad x\in[0,\ell],$$

with the homogeneous Dirichlet boundary condition: $u(0)=u(\ell)=0$. Clearly, $u(x)\equiv 0$ is a solution. Is this solution *unique or not*? Does the answer depend on ℓ?

Problem 5: Let $\Omega\in\mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Show that the following two *Neumann* problems are roughly equivalent (i.e., if you can find a solution to one of them, then you can also solve the other).

$$\begin{cases}\Delta v=f & \text{in } \Omega,\\ \dfrac{\partial v}{\partial \nu}=0 & \text{on } \partial\Omega.\end{cases}\tag{1}$$

$$\begin{cases}\Delta w=0 & \text{in } \Omega,\\ \dfrac{\partial w}{\partial \nu}=g & \text{on } \partial\Omega.\end{cases}\tag{2}$$

You can assume that there exists $\widetilde{g}\in C^2\left(\overline{\Omega}\right)$, an extension of g, such that $\dfrac{\partial\widetilde{g}}{\partial\nu}=g$ on $\partial\Omega$.

[Hint: Tailor what was done in class for the Dirichlet problems.]

Problem 6: Prove that the surface area and the volume of the unit ball $B_n(0,1) \in \mathbb{R}^n$ is

$$|\partial B_n(0,1)| =: \omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \quad |B_n(0,1)| = \frac{\omega_n}{n},$$

respectively.

Problem 7: Suppose $f(x) = \sum_{k=1}^{n} b_k \sin \pi k x$, $x \in [0,1]$, for some $b_k \in \mathbb{R}$, $k = 1, \dots, n$. Solve the following Dirichlet problem in the unit square $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u(x,0) = 0 & x \in [0,1] \\ u(0,y) = u(1,y) = 0 & y \in [0,1] \\ u(x,1) = f(x) & x \in [0,1]. \end{cases}$$