MAT 207B Methods of Applied Mathematics
Homework 5: due Friday, 02/14/25

Problem 1: Consider the piecewise constant function

$$f(x) = \begin{cases} 1 & \text{for } -\pi/2 \le x < \pi/2, \\ -1 & \text{for } \pi/2 \le x < 3\pi/2. \end{cases}$$

(a) Approximate f by a trigonometric polynomial consisting of five terms. Note that this turns out to be the same as truncating the Fourier series with $n = 2$, i.e., terms involving a_0, a_1, b_1, a_2, b_2 if they are all nonzeros. If some of them are zeros, you should find the first five nonzero such coefficients.

(b) Evaluate the error in at $x = 0$ and at $x = \pi/3$.

(c) Graph the function in (a) and the approximating polynomials consisting of the first term, the first three terms, and all five terms, using different colors, in the same coordinate system.

Problem 2: (a) Prove that

$$\int_0^\infty \frac{\sin(t)}{t}\,\mathrm{d}t = \pi/2.$$

(*Hint*: Consider the function $F(x) = \int_0^\infty \mathrm{e}^{-xt}\frac{\sin t}{t}\,\mathrm{d}t$, calculate $F'(x)$, integrate again and compute $F(0)$.)

(b) Transform

$$\int_{-h}^{h} \frac{\sin\left((n+\frac{1}{2})\xi\right)}{\sin\left(\frac{\xi}{2}\right)}\,\mathrm{d}\xi$$

by using the identity

$$\sin\left((n+\frac{1}{2})\xi\right) = \sin(n\xi)\cos(\xi/2) + \cos(n\xi)\sin(\xi/2) \tag{1}$$

and approximate $\sin(\xi/2)$ by $\xi/2$ and $\cos(\xi/2)$ by 1, assuming that h is very small. Show that in case these approximations are justified,

$$\lim_{n\to\infty}\int_{-h}^{h} \frac{\sin\left((n+\frac{1}{2})\xi\right)}{\sin\left(\frac{\xi}{2}\right)}\,\mathrm{d}\xi = 2\pi$$

by making use of the result in part (a).

Problem 3: Show that the following identities hold:

(a)

$$\frac{a_0^2\pi}{2} - a_0 \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = -\frac{a_0^2\pi}{2}$$

(b)

$$\int_{-\pi}^{\pi} \left(\sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx) \right)^2 \mathrm{d}x = \pi \sum_{k=1}^{n} (a_k^2 + b_k^2)$$

(c)

$$a_0 \int_{-\pi}^{\pi} \left(\sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx) \right) \mathrm{d}x = 0.$$

(d)

$$\int_{-\pi}^{\pi} f(x) \left(\sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx) \right) \mathrm{d}x = \pi \sum_{k=1}^{n} (a_k^2 + b_k^2).$$

Problem 4: Find the Fourier expansions for the following functions f_k, where $f_k(x) = f_k(x+2\pi)$:

(a) $f_1(x) = x$, $-\pi \le x < \pi$ (saw-tooth function)

(b) $f_2(x) = \begin{cases} x & -\pi/2 \le x < \pi/2 \\ -x+\pi & \pi/2 \le x < 3\pi/2 \end{cases}$ (zig-zag function)

(c) $f_3(x) = \begin{cases} -x(x-\pi) & 0 \le x < \pi \\ (x-\pi)(x-2\pi) & \pi \le x < 2\pi \end{cases}$

(d) $f_4(x) = \begin{cases} x^2(x-\pi)^2 & 0 \le x < \pi \\ (x-\pi)^2(x-2\pi)^2 & \pi \le x < 2\pi \end{cases}$

Problem 5: Let $f(\theta)$ be 2π-periodic and $f(\theta) = \theta^2$ for $-\pi < \theta < \pi$.

(a) Find its Fourier series.

(b) Using the result of Part (a) and the theorem of the Fourier coefficients of an integral of f *twice*, prove

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Problem 6: Let $f(\theta)$ be 2π-periodic and $f(\theta) = \mathrm{e}^{\theta}$ for $-\pi < \theta \le \pi$. Let $\sum_{-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}n\theta}$ be its Fourier series; thus $\mathrm{e}^{\theta} = \sum_{-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}n\theta}$ for $|\theta| < \pi$. If we formally differentiate this equation, we obtain $\mathrm{e}^{\theta} = \sum_{-\infty}^{\infty} \mathrm{i}nc_n \mathrm{e}^{\mathrm{i}n\theta}$. But then $c_n = \mathrm{i}nc_n$, or $(1-\mathrm{i}n)c_n = 0$, i.e., $c_n = 0$ for all $n \in \mathbb{Z}$. This is obviously wrong! Where is the mistake? Explain in details.