MAT 207B Methods of Applied Mathematics
Homework 6: due Friday, 02/21/25

Problem 1: Consider a differentiable function f on an interval $[a,b]$.

(a) Show that if $f'(x)$ is bounded on $[a,b]$, then $f \in BV[a,b]$.

[Hint: Show first that such f is *Lipschitz* continuous, i.e., $|f(x) - f(y)| \le L|x-y|$ for any $x, y \in [a,b]$ for some $L > 0$. Then, use the definition of the bounded variation for f to conclude the statement.]

(b) Let $f(x) = x^2 \sin(x^{-1})$ and $g(x) = x\sin(x^{-1})$ for $x \neq 0$, and $f(0) = g(0) = 0$. Show that $f \in BV[0,1]$ but $g \notin BV[0,1]$.

[Hint: Use $x_j = \left(\frac{\pi}{2} + j\pi\right)^{-1}$, $j = 0, 1, 2, \ldots$ as a partition of $[0,1]$ to show $g \notin BV[0,1]$.]

Problem 2: Find both the Fourier *cosine* series and the Fourier *sine* series of the following functions on the interval $[0,\pi]$. Then, answer what values these series converge when $\theta = 0$ and $\theta = \pi$.

(a) $f(\theta) = 1$.

(b) $f(\theta) = \pi - \theta$.

(c) $f(\theta) = \theta^2$.

Problem 3: Find the Fourier *sine* series on the interval $[0,\ell]$ of $f(x) := \ell x - x^2$.

Problem 4: Suppose f is a piecewise continuous function on $[0,\pi]$ such that $f(\theta) = f(\pi - \theta)$. Let a_n and b_n be the Fourier cosine and sine coefficients of f, respectively. Show that $a_n = 0$ for n odd and $b_n = 0$ for n even.

Problem 5: Show that the space $C[a,b]$ (the space of complex-valued continuous functions defined on the interval $[a,b]$) is *complete* with respect to the *supremum* norm $\|f\|_\infty := \sup_{x\in[a,b]} |f(x)|$.

Problem 6: Show that $\big|\|f\| - \|g\|\big| \le \|f - g\|$. Then, deduce that if $f_n \to f$ in norm, then $\|f_n\| \to \|f\|$.

Problem 7: Show that if $f_n \in L^2(a,b)$ and $f_n \to f$ in norm, then $\langle f_n, g\rangle \to \langle f, g\rangle$ for all $g \in L^2(a,b)$.

[Hint: Apply the Cauchy-Schwarz inequality to $\langle f_n - f, g\rangle$.]