MAT 207B Methods of Applied Mathematics
Homework 7: due Friday, 02/28/25

Problem 1: Suppose $\{\phi_n\}$ be an ONB for $L^2(a,b)$. Show that for any $f, g \in L^2(a,b)$,

$$\langle f, g\rangle = \sum_n \langle f, \phi_n\rangle \overline{\langle g, \phi_n\rangle},$$

which is also referred to as *Parseval's equality*.

Problem 2: For which $\alpha \in \mathbb{R}$ does the function $f_\alpha(x) := x^\alpha e^{-x}$ belong to $L^2(0,\infty)$? What is $\|f_\alpha\|$ when defined?

Problem 3: Show that $\left\{\sqrt{\frac{2}{\ell}}\cos\left(\left(n-\frac{1}{2}\right)\frac{\pi x}{\ell}\right)\right\}_{n=1}^{\infty}$ is an ONB for $L^2(0,\ell)$.

Problem 4: The *Legendre polynomial* of degree n is defined as

$$P_n(x) := \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n, \quad n = 0, 1, \ldots$$

Let $\varphi_n(x) := \sqrt{n+\frac{1}{2}}P_n(x)$. Show that $\{\varphi_n\}_{n=0}^{\infty}$ form an ONB for $L^2(-1,1)$.

[Hint: To show the *completeness*, consider $f \in L^2(-1,1)$ that is orthogonal to all the φ_n's. That implies that f is orthogonal to every polynomial. Then, the Weierstrass approximation theorem comes to your rescue.]

Problem 5: Let $P_n(x)$ be the Legendre polynomial of order n.

(a) Prove that for all $n \geq 1$,

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x),$$

and consequently,

$$\int P_n(x)\,\mathrm{d}x = \frac{1}{2n+1}[P_{n+1}(x) - P_{n-1}(x)] + C,$$

where C is an integration constant.

(b) Let

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } -1 \leq x \leq 0. \end{cases}$$

Expand f in a series of Legendre polynomials.

[Hint: Use the consequence of Part (a) as well as the following facts:

$$P_{2k-1}(0) = 0; P_{2k}(0) = \frac{(-1)^k (2k)!}{2^{2k}(k!)^2}.$$

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Problem 6: Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an ONB for $L^2(a,b)$, and let $f = \phi_1 + 2\phi_{100} + 2\phi_{101}$.
Let $S = \text{span}\{\phi_1, \ldots, \phi_{100}\} \subset L^2(a,b)$ be a linear span of the first 100 basis functions (i.e., all possible linear combinations of $\phi_1, \ldots, \phi_{100}$).

(a) Find the best *linear* approximation $\tilde{f} \in S$ to f in the sense of L^2 norm. Then, compute $\|\tilde{f} - f\|$.

(b) Find the best two-term *nonlinear* approximation $f_2 \in L^2(a,b)$ to f in the sense of L^2 norm. Then, compute $\|f_2 - f\|$.