MAT 207B Methods of Applied Mathematics
Homework 9 (The Last One!): due Friday, 03/14/25

Problem 1: The integral operator $\mathscr{K}$ on $L^2(0,\infty)$ is defined by

$$\mathscr{K} f(x) := \int_0^\infty k(x+\tau) f(\tau)\,\mathrm{d}\tau,$$

where $k \in C(0,\infty)$ is a complex-valued function such that

$$M := \int_0^\infty \tau |k(\tau)|^2\,\mathrm{d}\tau < \infty.$$

Show that $\mathscr{K}$ is Hilbert-Schmidt with Hilbert-Schmidt norm $\sqrt{M}$. Note that the *Hilbert-Schmidt norm* of a general integral operator $\mathscr{K} : L^2(a,b) \to L^2(c,d)$ with its kernel $k(\cdot,\cdot) \in L^2((c,d)\times(a,b))$ is defined as

$$\|\mathscr{K}\|_{\mathrm{HS}} := \sqrt{\int_c^d \int_a^b |k(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau}.$$

Problem 2: By applying the Sturm-Liouville Theorem to the system

$$\begin{cases} f'' + \lambda f = 0, & 0 \le x \le \pi; \\ f(0) = 0 = f'(\pi), \end{cases}$$

show that for any $g \in L^2(0,\pi)$,

$$g(x) = \sum_{j=1}^\infty c_j \sin\left(j - \frac{1}{2}\right) x$$

in the norm of $L^2(0,\pi)$ where

$$c_j = \frac{2}{\pi} \int_0^\pi g(x) \sin\left(j - \frac{1}{2}\right) x\,\mathrm{d}x, \quad j \in \mathbb{N}.$$

Problem 3: Assume the following fact: $\{\phi_k(x) := J_0(z_k x/b)\}_{k\in\mathbb{N}}$ *forms an* orthogonal basis *for* $L^2_w(0,b)$ *with* $w(x) = x$ *and* z_k *as the kth zero of* $J_0(x)$*, and* $\|\phi_k\|_w^2 = \frac{b^2}{2} J_1(z_k)^2$ *where* J_1 *is the Bessel function of the first kind of order* 1. Then, expand the characteristic function $\chi_{[0,b/2]}(x)$ as a *Fourier-Bessel* series $\sum_{k=1}^\infty c_k J_0(z_k x/b)$; that is, compute the coefficients c_k, $k \in \mathbb{N}$.

[Hint: You may want to use the following famous formula:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x), \quad \forall x \in \mathbb{R}; \forall \nu \in \mathbb{R}.$$

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Problem 4: Make the solution of the hanging chain problem given in the last theorem in Lecture 25 *explicit* by using the expressions for λ_j and φ_j in terms of J_0, i.e., the Bessel function of the first kind of order 0 and its zeros z_j.