<u>A Caution on the Alternating Series Test</u>

Theorem 14 (The Alternating Series Test) of the textbook says:
The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if ***all*** *of the following conditions are satisfied:*

1. $u_n > 0$ *for all* $n \in \mathbb{N}$.
2. $u_n \geq u_{n+1}$ *for all* $n \geq N$, *for some integer* N.
3. $u_n \to 0$ *as* $n \to \infty$.

Clearly, to show the convergence, you need to check all these three conditions. What can you conclude if a given alternating series fails one or more of the three conditions? Is that series divergent? Unfortunately, the failure of this test does not immediately lead to the divergence! You need to use some other test, often, the ***n*th Term Test for Divergence** to conclude that the series diverges. The reason is the following. The equivalent statement to Theorem 14 is that:

(1) *"If a given alternating series diverges, then it fails to satisfy one or more of the above three conditions."*

This is different from the following statement, which is **false**:

(2) *"If a given alternating series fails to satisfy one or more of the above three conditions, then the series diverges."*

We need to realize the basic logic here: The **contraposition** of *"If A is true, then B is true."* is *"If B is false, then A is false."* These two statements are equivalent. So, the contraposition of Theorem 14 is the statement (1) above, not (2).

Let's work out an example, which demonstrates that we need some care if the Alternating Series Test fails. Consider the following alternating series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2n-1}.$$

So, $u_n = \dfrac{n}{2n-1}$. Condition 1 in Theorem 14 holds. Also, Condition 2, i.e., $u_n > u_{n+1}$ holds since

$$u_n - u_{n+1} = \frac{n}{2n-1} - \frac{n+1}{2n+1} = \frac{n(2n+1)-(n+1)(2n-1)}{(2n-1)(2n+1)} = \frac{1}{(2n-1)(2n+1)} > 0.$$

However, Condition 3 clearly fails because $\lim_{n\to\infty} u_n = \frac{1}{2} \neq 0$. Hence, we cannot declare the convergence of this series using the Alternating Series Test. Is this series divergent? Yes, it is divergent since $\lim_{n\to\infty} (-1)^{n+1}u_n$ does not exist (i.e., the nth Term Test for Divergence holds). This is a correct reasoning to show the divergence of the above series. In fact, in this example, it would be much easier and simpler to use the nth Term Test of Divergence from the start without referring the Alternating Series Test. **So here is a good way of testing a given alternating series: if you see the alternating series, check first the nth Term Test for Divergence (i.e., check if $\lim_{n\to\infty} (-1)^{n+1}u_n$ does not exist or converge to a non-zero value). If this test holds, then the series diverges and it's the end of the story. If not, in other words, if $\lim_{n\to\infty} (-1)^{n+1}u_n = 0$, then apply the Alternating Series Test to check the convergence.** This way, you can avoid unnecessary computation and explanation since the nth Term Test for Divergence is less time consuming than the Alternating Series Test.