Problem 1 (10 pts) Find the radius and the interval of convergence of the series

\[ \sum_{n=0}^{\infty} \frac{(2x-1)^n}{\sqrt{n+3}}. \]

Solution: Let \( a_n = \frac{(2x-1)^n}{\sqrt{n+3}} \). The absolute ratio test gives us:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(2x-1)^{n+1}}{\sqrt{n+4}} \cdot \frac{\sqrt{n+3}}{(2x-1)^n} \right|
= \lim_{n \to \infty} \sqrt{\frac{n+3}{n+4}} |2x-1|
= |2x-1|.
\]

Hence, for this power series to converge absolutely, we must have \(|2x-1| < 1\), i.e., \( |x - \frac{1}{2}| < \frac{1}{2} \) or \( 0 < x < 1 \). So, the radius of convergence is \( R = \frac{1}{2} \). Now, to determine the interval of convergence, we need to check the convergence at the boundary points, \( x = 0, 1 \).

For \( x = 0 \), we have \( \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}. \) This is an alternating series and it satisfies the Alternating Series Test:

(i) \( u_n = \frac{1}{\sqrt{n+3}} > 0 \) for all \( n \in \mathbb{N} \);

(ii) \( u_{n+1} = \frac{1}{\sqrt{n+4}} < \frac{1}{\sqrt{n+3}} = u_n \) for all \( n \in \mathbb{N} \);

(iii) \( u_n \to 0 \) as \( n \to \infty \).

Hence, this alternating series converges conditionally.

On the other hand, for \( x = 1 \), we have

\[ \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}} = \sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}, \]

which is a \( p \)-series with \( p = \frac{1}{2} < 1 \). Hence this series diverges.

Therefore, the interval of convergence of the given series is:

\[ 0 \leq x < 1 \quad \text{or} \quad x \in (0, 1). \]

Score of this page:_______________
Problem 2 (10 pts) Consider the function \( f(x) = \sqrt{1 + x} \).

(a) (5 pts) Approximate \( f(x) \) by a Taylor polynomial of degree 1 centered at 0. Compute the value of \( \sqrt{2} \) using that approximation.

(b) (5 pts) Compute the worst-case approximation error for \( 0 \leq x \leq 1 \). Then, confirm that the approximate value of \( \sqrt{2} \) computed in Part (a) is within this worst-case error. Note that the precise value of \( \sqrt{2} \) in 3 decimal places is 1.414.

Solution to (a): First of all, since \( f(x) = (1 + x)^{1/2} \), we have \( f'(x) = \frac{1}{2}(1 + x)^{-1/2} \), and \( f''(x) = -\frac{1}{4}(1 + x)^{-3/2} \). Hence, by Taylor's formula, we have

\[
\sqrt{1 + x} = P_1(x) + R_1(x)
\]

\[
= f(0) + f'(0)x + \frac{f''(c)}{2}x^2 \quad 0 < c < x \text{ or } x < c < 0
\]

\[
= 1 + \frac{1}{2}x - \frac{1}{8}(1 + c)^{-3/2}x^2.
\]

Hence, the Taylor polynomial of degree 1 is: \( P_1(x) = 1 + \frac{1}{2}x \). So, the approximate value of \( \sqrt{2} \) is simply \( \sqrt{2} \approx P(1) = 1.5 \).

Solution to (b): From Taylor's formula, we have

\[
\left| f(x) - P_1(x) \right| = \left| R_1(x) \right| = \frac{1}{8} \left| \frac{x^2}{(1 + c)^{3/2}} \right|
\]

\[
\leq \frac{1}{8 \cdot (1 + c)^{3/2}} \quad \text{since } 0 \leq x \leq 1.
\]

\[
< \frac{1}{8} \quad \text{since } 0 < c < 1.
\]

Hence the approximation error is bounded by \( \frac{1}{8} = 0.125 \), which is the worst-case scenario. Now, \( f(1) = \sqrt{2} \), so \( |f(1) - P_1(1)| = |1.414 - 1.5| = 0.086 < 0.125 \). So, this is within the error bound.

Score of this page:___________
Problem 3 (10 pts)

(a) (5 pts) Write \( \text{proj}_v u \), i.e., the vector projection of \( u \) onto \( v \), using the dot product between \( u \) and \( v \), the length of the vector \( v \), and the vector \( v \) itself.

(b) (5 pts) Let \( u = j + k \), \( v = i + j \). Then write \( u \) as the sum of a vector parallel to \( v \) and a vector orthogonal to \( v \).

Solution to (a): The vector projection of \( u \) onto \( v \) is proportional (i.e., parallel) to \( v \) by definition. Let \( \theta \) be an angle between \( u \) and \( v \). Then, from the geometry of these two vectors, we have

\[
\text{proj}_v u = \left( |u| \cos \theta \right) \frac{v}{|v|} = \left( |u| |v| \cos \theta \right) \frac{v}{|v|^2} = \frac{(u \cdot v)}{|v|^2} v \text{ or equivalently } \frac{u \cdot v}{|v|^2} v.
\]

Solution to (b): From the geometry and the definition of the vector projection, it is clear that \( \text{proj}_v u \) is parallel to \( v \) and perpendicular to the residual \( u - \text{proj}_v u \). Hence,

\[
\text{proj}_v u = \frac{(j + k) \cdot (i + j)}{|i + j|^2} (i + j) = \frac{\langle 0, 1, 1 \rangle \cdot \langle 1, 1, 0 \rangle}{|\langle 1, 1, 0 \rangle|^2} \langle 1, 1, 0 \rangle = \frac{1}{2} \langle 1, 1, 0 \rangle = \langle \frac{1}{2}, \frac{1}{2}, 0 \rangle.
\]

Hence, the residual is

\[
u - \text{proj}_v u = \langle 0, 1, 1 \rangle - \langle \frac{1}{2}, \frac{1}{2}, 0 \rangle = \langle -\frac{1}{2}, \frac{1}{2}, 1 \rangle.
\]

So, the vector \( u \) is written in the desired form as follows:

\[
u = j + k = \left( \frac{1}{2} i + \frac{1}{2} j \right) + \left( -\frac{1}{2} i + \frac{1}{2} j + k \right)
\]
or equivalently

\[
u = \langle 0, 1, 1 \rangle = \langle \frac{1}{2}, \frac{1}{2}, 0 \rangle + \langle -\frac{1}{2}, \frac{1}{2}, 1 \rangle.
\]
Problem 4  (10 pts)

Let $L$ be a line that passes through the points $Q$ and $R$. Let $P$ be a point not on the line $L$. Show that the distance $d$ from the point $P$ to the line $L$ is

$$d = \frac{|a \times b|}{|a|},$$

where $a = \overrightarrow{QR}$, $b = \overrightarrow{QP}$.

Hint: Drawing a figure will help.

Solution: Let $\theta$ be an angle between $a = \overrightarrow{QR}$ and $b = \overrightarrow{QP}$. Then, the distance $d$ can be easily computed by

$$d = |b||\sin \theta|. \quad (1)$$

On the other hand, the definition of the cross product gives us

$$a \times b = (|a||b|\sin \theta)n,$$

where $n$ is a unit vector perpendicular to both $a$ and $b$. Hence,

$$|a \times b| = |a||b||\sin \theta||n| = |a||b||\sin \theta| = |a|d,$$

where we used Eq. (1) to derive the last equality. So, we have:

$$d = \frac{|a \times b|}{|a|}. $$

Score of this page:_______________
Problem 5 (10 pts) Find the limit if it exists, or show that the limit does not exist.

(a) (5 pts)
\[ \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4}. \]
Hint: Consider \((x, y) \to (0, 0)\) along the line \(y = x\) and along the parabola \(y^2 = x\).

(b) (5 pts)
\[ \lim_{(x,y) \to (0,0)} \frac{x^3 + y^3}{x^2 + y^2}. \]
Hint: Consider the limit in the polar coordinates \((r, \theta)\).

Solution to (a): Suppose \((x, y) \to (0, 0)\) along the line \(y = x\). Then,
\[ \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4} = \lim_{y \to 0} \frac{y^3}{y^2 + y^4} = \lim_{y \to 0} \frac{y}{1 + y^2} = 0. \]
However, along the parabola \(y^2 = x\), we have
\[ \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4} = \lim_{y \to 0} \frac{y^4}{y^4 + y^4} = \lim_{y \to 0} \frac{1}{2} = \frac{1}{2}. \]
Of course you can get the same limit by going through \(x\) instead of \(y\), i.e.,
\[ \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4} = \lim_{x \to 0} \frac{x^2}{x^2 + x^2} = \frac{1}{2}. \]
Hence the value depends on how \((x, y)\) approaches \((0, 0)\). So, the limit does not exist.

Solution to (b): Consider the polar coordinates: \(x = r \cos \theta, y = r \sin \theta\), where \(r \geq 0, 0 \leq \theta < 2\pi\).
\[ \lim_{(x,y) \to (0,0)} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{r \to 0} \frac{r^3(\cos^3 \theta + \sin^3 \theta)}{r^2(\cos^2 \theta + \sin^2 \theta)} = \lim_{r \to 0} \frac{r(\cos^3 \theta + \sin^3 \theta)}{r} = (\cos^3 \theta + \sin^3 \theta) \lim_{r \to 0} r = 0. \]
This is true regardless of the direction of approach \(\theta\). Hence, the limit exists and its value is 0.
Problem 6 (10 pts) Verify that the function

\[ u(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \]

satisfies Laplace's equation

\[ u_{xx} + u_{yy} + u_{zz} = 0. \]

Hint: You may want to use the symmetry between \( x, y, z \) in this problem to save your computation of the partial derivatives.

Solution: This is a quite straightforward problem. Since \( \frac{1}{\sqrt{x^2 + y^2 + z^2}} = (x^2 + y^2 + z^2)^{-\frac{1}{2}} \), we have

\[ u_x = \frac{\partial u}{\partial x} = -\frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{3}{2}} \cdot 2x = -x (x^2 + y^2 + z^2)^{-\frac{3}{2}}. \]

So,

\[ u_{xx} = \frac{\partial^2 u}{\partial x^2} = -(x^2 + y^2 + z^2)^{-\frac{3}{2}} - x \frac{-3}{2} (x^2 + y^2 + z^2)^{-\frac{5}{2}} \cdot 2x = (2x^2 - y^2 - z^2) \cdot (x^2 + y^2 + z^2)^{-\frac{5}{2}}. \]

Since \( u(x, y, z) = (x^2 + y^2 + z^2)^{-\frac{1}{2}} \) is symmetric with respect to \( x, y, z \), we immediately have

\[ u_{yy} = \frac{\partial^2 u}{\partial y^2} = (2y^2 - z^2 - x^2) \cdot (x^2 + y^2 + z^2)^{-\frac{5}{2}}, \quad u_{zz} = \frac{\partial^2 u}{\partial z^2} = (2z^2 - x^2 - y^2) \cdot (x^2 + y^2 + z^2)^{-\frac{5}{2}}. \]

Hence,

\[ u_{xx} + u_{yy} + u_{zz} = (2x^2 - y^2 - z^2 + 2y^2 - z^2 - x^2 + 2z^2 - x^2 - y^2) \cdot (x^2 + y^2 + z^2)^{-\frac{5}{2}} = 0 \cdot (x^2 + y^2 + z^2)^{-\frac{5}{2}} = 0. \]
Problem 7 (10 pts) Assuming that the equation
\[ xe^y + \sin(xy) + y - \ln 2 = 0 \]
defines \( y \) as a differentiable function of \( x \), use the Implicit Differentiation Theorem to find the value of \( \frac{dy}{dx} \) at the point \((x, y) = (0, \ln 2)\).

Solution: Let \( F(x, y) = xe^y + \sin(xy) + y - \ln 2 = 0 \). Then, the Implicit Differentiation Theorem says that
\[ \frac{dy}{dx} = -\frac{F_x}{F_y}. \]
Now,
\[ F_x(x, y) = e^y + y \cos(xy), \quad F_y(x, y) = xe^y + x \cos(xy) + 1. \]
Hence,
\[ F_x(0, \ln 2) = e^{\ln 2} + \ln 2 \cdot \cos(0) = 2 + \ln 2, \quad F_y(0, \ln 2) = 0 + 0 + 1 = 1. \]
Hence,
\[ \left. \frac{dy}{dx} \right|_{(x, y) = (0, \ln 2)} = -\frac{2 + \ln 2}{1} = -(2 + \ln 2). \]
Problem 8  (10 pts) Find the direction in which $f(x, y) = \sin x + e^{xy}$

(a) (3 pts) Increases most rapidly at the point $(0, 1)$,

(b) (3 pts) Decreases most rapidly at the point $(0, 1)$.

(c) (4 pts) Does not change (i.e., is flat) at the point $(0, 1)$.

Solution to (a): The gradient vector is

$$\nabla f(x, y) = \langle f_x, f_y, \rangle = \langle \cos x + ye^{xy}, xe^{xy} \rangle.$$

So, $\nabla f(0, 1) = \langle 2, 0 \rangle$. Now, $f$ increases most rapidly in the direction of $\nabla f$. Hence, this direction is $\langle 2, 0 \rangle$. You can also say the unit vector $\langle 1, 0 \rangle$ is the direction.

Solution to (b): $f$ decreases most rapidly in the direction of $-\nabla f$. Hence, it is $\langle -2, 0 \rangle$. Also fine is $\langle -1, 0 \rangle$.

Solution to (c): $f$ does not change in the direction orthogonal to $\nabla f$. Let $u = \langle u_1, u_2 \rangle$ be this directional vector. Then,

$$u \cdot \nabla f(0, 1) = 0 \iff u = \langle u_1, u_2 \rangle \cdot \langle 2, 0 \rangle = 0 \iff 2u_1 = 0.$$

So, $u_1$ must be zero, but $u_2$ can be any nonzero value, say $u_2 = 1$.

Hence, along the direction $u = \langle 0, 1 \rangle$, this function $f$ does not change (or is flat) at the point $(0, 1)$.

You can also conclude immediately (rather than going through the above computation) that the orthogonal direction to the vector $\langle 2, 0 \rangle$ or $\langle 1, 0 \rangle$ is $\langle 0, 1 \rangle$.

(Of course, the vector $u = \langle 0, -1 \rangle$ is fine too instead of $\langle 0, 1 \rangle$).
Problem 9  (10 pts) Consider the sphere with radius $r > 0$ in 3D, $x^2 + y^2 + z^2 = r^2$.

(a)  (5 pts) Find the tangent plane at the point $\left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right)$ on this sphere.

(b)  (5 pts) Show that every normal line to this sphere passes through the center of the sphere, i.e., the origin.

Hint: Pick any point $(x_0, y_0, z_0)$ on this sphere, and consider the normal line at that point.

Solution to (a): Let $F(x, y, z) = x^2 + y^2 + z^2 - r^2 = 0$. The equation of the tangent plane to $F$ at $(x_0, y_0, z_0)$ is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Since we have

$$F_x(x, y, z) = 2x, \quad F_y(x, y, z) = 2y, \quad F_z(x, y, z) = 2z,$$

the equation of the tangent plane becomes

$$\frac{2r}{\sqrt{3}} \left( x - \frac{r}{\sqrt{3}} \right) + \frac{2r}{\sqrt{3}} \left( y - \frac{r}{\sqrt{3}} \right) + \frac{2r}{\sqrt{3}} \left( z - \frac{r}{\sqrt{3}} \right) = 0 \iff x + y + z = \sqrt{3} r.$$

Solution to (b): At the point $(x_0, y_0, z_0)$, the gradient vector is $\nabla F(x_0, y_0, z_0) = (2x_0, 2y_0, 2z_0)$. The normal line passes through the point $(x_0, y_0, z_0)$ is parallel to this gradient vector. Hence, the equation of this normal line is

$$\begin{cases}
    x = x_0 + t \cdot 2x_0 = x_0(1 + 2t) \\
    y = y_0 + t \cdot 2y_0 = y_0(1 + 2t) \\
    z = z_0 + t \cdot 2z_0 = z_0(1 + 2t)
\end{cases},$$

where $t$ is an arbitrary real-valued parameter. Hence, for any point $(x_0, y_0, z_0)$ on the sphere, $t = -\frac{1}{2}$ gives us the common point $(0, 0, 0)$, i.e., the center of this sphere. In other words, any normal line of the sphere always passes through its center.

Score of this page:_______________
Problem 10 (10 pts) Let \( f(x, y) = 2x^2 + y^2 \).

(a) (5 pts) Find the linearization at the point \((1, 1)\). Then use it to approximate \( f(0.9, 1.1) \). Compare the approximate value with the true value.

(b) (5 pts) Approximate \( f(2, 2) \) using the same linearization. Compare the approximate value with the true value. At which point is the linear approximation better, \((0.9, 1.1)\) or \((2, 2)\)?

Solution to (a): The linearization of \( f(x, y) \) at the point \((x_0, y_0)\) is

\[
L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

We have

\[
f_x(x, y) = 4x, \quad f_y(x, y) = 2y.
\]

At the point \((1, 1)\), we have \( f(1, 1) = 3 \) as well as

\[
f_x(1, 1) = 4, \quad f_y(1, 1) = 2.
\]

Hence,

\[
L(x, y) = 3 + 4(x - 1) + 2(y - 1) = 4x + 2y - 3.
\]

Let's now use this to approximate \( f(0.9, 1.1) \).

\[
L(0.9, 1.1) = 4 \times 0.9 + 2 \times 1.1 - 3 = 2.8.
\]

On the other hand, the true value is:

\[
f(0.9, 1.1) = 2 \times 0.9^2 + 1.1^2 = 1.62 + 1.21 = 2.83.
\]

So, the approximation is quite close to the true value and the error is

\[
|f(0.9, 1.1) - L(0.9, 1.1)| = 0.03.
\]

Solution to (b): We have

\[
L(2, 2) = 4 \cdot 2 + 2 \cdot 2 - 3 = 9.
\]

\[
f(2, 2) = 2 \cdot 2^2 + 2^2 = 12.
\]

Hence, the approximation error is: \(|f(2, 2) - L(2, 2)| = 3\). This error is much worse than that at \((0.9, 1.1)\), which is 0.03.

Hence, the linear approximation at \((0.9, 1.1)\) is better than that at \((2, 2)\). The reason is that the point \((0.9, 1.1)\) is much closer to the tangential point \((1, 1)\) than the point \((2, 2)\) is.

Score of this page:_______________
Problem 11 (10 pts) Consider the following function over the closed domain \( \Omega = \{(x, y) \mid -1 \leq x \leq 1, -1 \leq y \leq 1\} \):

\[
 f(x, y) = x \cos y.
\]

(a) (5 pts) Find the local maxima, local minima, saddle points of \( f \) if any.

(b) (5 pts) Find the absolute maxima and absolute minima.

Solution to (a): Let’s compute all the critical points in \( \Omega \).

Since \( f_x(x, y) = \cos y \) and \( f_y(x, y) = -x \sin y \), setting these two equations to 0 gives us \( \cos y = 0 \) and \( -x \sin y = 0 \). From these two, we must have \( x = 0 \) and \( \cos y = 0 \). Because \(-\pi \leq y \leq \pi\), \( \cos y = 0 \) implies \( y = \pm \frac{\pi}{2} \). Hence, the critical points are \( \left(0, \pm \frac{\pi}{2}\right) \).

Now, let’s do the Second Derivative Test. First, we compute the 2nd partial derivatives:

\[
 f_{xx}(x, y) = 0, \quad f_{yy}(x, y) = -x \cos y, \quad f_{xy}(x, y) = -\sin y.
\]

From these, we have

\[
 D(0, \frac{\pi}{2}) = f_{xx}(0, \frac{\pi}{2}) f_{yy}(0, \frac{\pi}{2}) - f_{xy}(0, \frac{\pi}{2})^2 = 0 - (-1)^2 = -1 < 0.
\]

Similarly, we get \( D(0, -\frac{\pi}{2}) = -1 < 0 \).

Hence, our conclusion:

There are two saddle points \( f(0, \frac{\pi}{2}) = f(0, -\frac{\pi}{2}) = 0 \), but neither local minimum nor local maximum exist.

Solution to (b): Now we need to check the extreme values on the boundary of \( \Omega \).

On \( y = -\pi \): \( f(x, -\pi) = x \cos(-\pi) = -x, -\pi \leq x \leq \pi \). So, on this boundary, the minimum is \( f(\pi, -\pi) = -\pi \) while the maximum is \( f(-\pi, -\pi) = \pi \).

On \( x = \pi \): \( f(\pi, y) = \pi \cos y, -\pi \leq y \leq \pi \). So, on this boundary, the minimum is \( f(\pi, -\pi) = -\pi \) while the maximum is \( f(\pi, 0) = \pi \).

On \( y = \pi \): \( f(x, \pi) = x \cos(\pi) = -x, -\pi \leq x \leq \pi \). So, on this boundary, the minimum is \( f(\pi, \pi) = -\pi \) while the maximum is \( f(-\pi, \pi) = \pi \).

On \( x = -\pi \): \( f(-\pi, y) = -\pi \cos y, -\pi \leq y \leq \pi \). So, on this boundary, the minimum is \( f(-\pi, 0) = -\pi \) while the maximum is \( f(-\pi, \pi) = \pi \).

Combining all the results we got so far, we can easily conclude that

the absolute maximum is \( f(\pi, 0) = f(-\pi, -\pi) = f(-\pi, \pi) = \pi \) while the absolute minimum is \( f(-\pi, 0) = f(\pi, -\pi) = f(\pi, \pi) = -\pi \).
Problem 12 (10 pts) Use Lagrange multipliers to find the maximum and minimum values of the function
\[ f(x, y) = xy \quad \text{subject to} \quad x^2 + y^2 = 1. \]

Note that we only consider the real values for \( x \) and \( y \), not the complex values.

Solution: Let \( g(x, y) = x^2 + y^2 - 1 = 0 \) be the constraint. Let's find the points \((x, y)\) and the parameter \( \lambda \) satisfying
\[ \nabla f = \lambda \nabla g, \quad g(x, y) = 0. \tag{2} \]

We have
\[ \nabla f = \langle f_x, f_y \rangle = \langle y, x \rangle, \quad \nabla g = \langle g_x, g_y \rangle = \langle 2x, 2y \rangle. \]

Hence, Eq. (2) leads to
\[ \begin{align*}
y &= 2\lambda x \tag{3} \\
x &= 2\lambda y \tag{4} \\
x^2 + y^2 &= 1. \tag{5}
\end{align*} \]

We can see that \( \lambda \neq 0 \) because if \( \lambda = 0 \), then \( x = y = 0 \) from Eqn's (3) and (4), which contradicts Eq. (5). Similarly, \( x \neq 0 \) as well as \( y \neq 0 \). Hence, we can divide Eq. (3) by Eq. (4) to get
\[ \frac{y}{x} = \frac{2\lambda x}{2\lambda y} \iff \frac{y}{x} = \frac{x}{y} \iff x^2 = y^2. \]

This implies \( x = y \) or \( x = -y \). Insert these into Eq. (5) gives us \( 2x^2 = 1 \) or \( x = \pm \frac{1}{\sqrt{2}} \). So, we have \((x, y) = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right), \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \text{ or } \left( -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \). From Eq. (3), the corresponding \( \lambda \)'s can be easily calculated as \( \lambda = \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \text{ or } \frac{1}{2} \), respectively. Hence,
\[ \begin{align*}
f\left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) &= f\left( -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) = \frac{1}{2}, \\
f\left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) &= f\left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = -\frac{1}{2}.
\end{align*} \]

Therefore, the maxima are
\[ f\left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = f\left( -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) = \frac{1}{2}, \]
and the minima are
\[ f\left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) = f\left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = -\frac{1}{2}. \]

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