Problem 1 (15 pts) Does the following sequence converge or diverge as \( n \to \infty \)? Give reasons for your answer. If it converges, find the limit.

(a) (7 pts)

\[ a_n = n^2 e^{-n} \]

**Answer:** Let us define the function \( f(x) = x^2 e^{-x} \) for all \( x \geq 1 \). If \( \lim_{x \to \infty} f(x) \) exists, then

\[ \lim_{n \to \infty} f(n) = \lim_{x \to \infty} f(x) \].

Now,

\[
\lim_{x \to \infty} x^2 e^{-x} = \lim_{x \to \infty} \frac{x^2}{e^x}
\]

\[ = \lim_{x \to \infty} \frac{2x}{e^x} \text{ by l'Hôpital's rule.} \]

\[ = \lim_{x \to \infty} \frac{2}{e^x} \text{ by l'Hôpital's rule again.} \]

\[ = 0. \]

Therefore, this sequence converges to the limit 0.

(b) (8 pts)

\[ a_n = n \cos \frac{1}{n} \]

[ Hint: Consider how the graph of \( \cos x \) behave near \( x = 0 \). You may also want to use the fact: \( \cos 1 \approx 0.54 \). ]

**Answer:** It diverges. Notice that \( \cos \frac{1}{n} > \cos 1 \approx 0.54 \) for \( n = 2, 3, \ldots \). Hence we have:

\[ n \cos \frac{1}{n} > 0.54n \text{ for } n = 2, 3, \ldots \]

Now, the righthand side tends to \( \infty \) as \( n \to \infty \). Therefore, the lefthand side must go to \( \infty \), i.e.,

\[ \lim_{n \to \infty} n \cos \frac{1}{n} = \infty. \]
Problem 2  (20 pts) Does the following series absolutely converge, conditionally converge, or diverge? Give reasons for your answer.

\[ \sum_{n=1}^{\infty} \left( 1 - \frac{1}{n} \right)^n \]

[ Hint: You may want to use the following formula for a particular value of \( x \):

\[ \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = e^x \quad \forall x \in \mathbb{R}. \]

Also, you may want to use the fact: \( e^{-1} \approx 0.36788 \). ]

Answer: Let \( a_n = \left( 1 - \frac{1}{n} \right)^n \). Note that \( a_n \geq 0 \) for every \( n \in \mathbb{N} \). Hence, there is no need to check the absolute convergence nor the conditional convergence. Simply checking the usual convergence suffice. Now, we will use the Root Test.

\[ \sqrt[n]{a_n} = \left( 1 - \frac{1}{n} \right)^n = \left( 1 + \frac{-1}{n} \right) ^n \to e^{-1} \approx 0.36788 \quad \text{as} \quad n \to \infty. \]

Hence, the series \( \sum_{n=1}^{\infty} a_n \) (absolutely) converges via the Root Test since \( \rho = e^{-1} < 1 \).

Score of this page:_______________
Problem 3 (20 pts) Does the following series absolutely converge, conditionally converge, or diverge? Give reasons for your answer.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

[ Hint: Use either the Comparison Test or the Integral Test. ]

Answer (via the Comparison Test): Let $$a_n = \frac{\ln n}{n}$$. It is clear that $$a_n \geq 0$$ for every $$n \in \mathbb{N}$$. So, we can use the Comparison Test. Notice that $$\ln n > 1$$ for every $$n \geq 3$$ since $$e \approx 2.718$$. Therefore,

$$\frac{\ln n}{n} > \frac{1}{n}$$ for every $$n \geq 3$$.

The series $$\sum_{n=3}^{\infty} \frac{1}{n}$$ diverges because this is the harmonic series. Therefore, by the Comparison Test, the series $$\sum_{n=1}^{\infty} a_n$$ diverges so does $$\sum_{n=1}^{\infty} a_n$$.

Answer (via the Integral Test): Let $$f(x) = \frac{\ln x}{x} \geq 0$$ for $$x \geq 1$$. Then, $$f(n) = \frac{\ln n}{n} \geq 0$$ for every $$n \in \mathbb{N}$$. So, let us set $$a_n = f(n)$$. Hence, we can apply the Integral Test, i.e., $$\sum_{n=1}^{\infty} a_n$$ and $$\int_{1}^{\infty} f(x) \, dx$$ share the same fate. Now notice that using Integration by Parts, we have

$$\int \frac{\ln x}{x} \, dx = (\ln x)^2 - \int \frac{\ln x}{x} \, dx.$$ 

That is,

$$\int \frac{\ln x}{x} \, dx = \frac{1}{2}(\ln x)^2.$$ 

(You can check this is correct by differentiating the righthand side.) Hence,

$$\int_{1}^{\infty} \frac{\ln x}{x} \, dx = \left[ \frac{1}{2}(\ln x)^2 \right]_{1}^{\infty} = \infty.$$ 

That is, this integral diverges so does this series via the Integral Test.

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Problem 4 (20 pts) Determine the radius and the interval of convergence of the power series:

\[ f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} n(x - 1)^n. \]

Justify your answers.

Answer: Let \( a_n = (-1)^{n-1} n(x - 1)^n \). Then,

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^n (n+1)(x - 1)^{n+1}}{(-1)^{n-1} n(x - 1)^n} \right| = \frac{n+1}{n} |x - 1| \rightarrow |x - 1| \quad \text{as} \quad n \rightarrow \infty \quad \text{regardless of the value of} \quad x.
\]

Therefore, if \(|x - 1| < 1\), then this power series converges absolutely (and hence converges) by the Ratio Test. This means that the radius of convergence is \( R = 1 \).

As for the interval of convergence, we need to check the end points of the obvious interval \(-1 < x - 1 < 1\), i.e., \( 0 < x < 2 \). If \( x = 0 \), then \( f(0) = \sum_{n=1}^{\infty} (-1)^{2n-1} n = -\sum_{n=1}^{\infty} n \). The \( n \)th term of the series does not approach zero therefore the series diverges, specifically to \(-\infty\). Hence, \( x = 0 \) cannot be included in the interval of convergence. For \( x = 2 \), \( f(2) = \sum_{n=1}^{\infty} (-1)^{n-1} n \), which diverges because the \( n \)th term of the series does not approach zero. Hence, \( x = 2 \) cannot be included in the interval of convergence either. Therefore, the interval of convergence is \( 0 < x < 2 \), or \( x \in (0, 2) \).

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Problem 5  (25 pts) Let \( f(x) = \cos x \).

(a) (10 pts) Find the Maclaurin series for \( f(x) \).

Answer: Let \( f(x) = \cos x \). Then, we have the following derivatives: 
\[
\begin{align*}
  f^{(2k)}(x) &= (-1)^k \cos x, \\
  f^{(2k+1)}(x) &= (-1)^{k+1} \sin x, \\
  k &= 0, 1, \ldots
\end{align*}
\]

Hence, \( f^{(2k)}(0) = (-1)^k \) while \( f^{(2k+1)}(0) = 0 \).

Therefore, we have the following Taylor series of \( \cos x \) at \( x = 0 \):
\[
\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.
\]

(b) (10 pts) Suppose we want to approximate \( \cos x \) using \( P_2(x) \), i.e., the Taylor polynomial of order 2, centered at \( x = 0 \). Use the Remainder Estimation Theorem to determine the range of \( x \) if we want to keep the magnitude of error between \( \cos x \) and \( P_2(x) \) less than 0.0001, i.e., \( |\cos x - P_2(x)| < 0.0001 \).

[ Hint: You may want to use the fact: \((0.0006)^{1/3} \approx 0.0843 \). ]

Answer: First of all, the Taylor polynomial of order 2 is clearly
\[
P_2(x) = 1 - \frac{x^2}{2!}.
\]

Using Taylor’s formula, we have
\[
\cos x = P_2(x) + R_2(x) = 1 - \frac{x^2}{2!} + \frac{f^{(3)}(c)}{3!} x^3
\]
for some \( c \) between 0 and \( x \).

Since \( |f^{(3)}(c)| = |\sin c| \leq 1 \) for all values of \( c \). Thus by the Remainder Estimation Theorem
\[
|\cos x - P_2(x)| = |R_2(x)| \leq \frac{1}{3!} |x|^3.
\]

Now to determine the range of \( x \) values for which the magnitude of error between \( \cos x \) and \( P_2(x) \) less than 0.0001, we find \( x \) such that
\[
|R_2(x)| \leq \frac{1}{3!} |x|^3 < 0.0001 \iff \frac{1}{3!} |x|^3 < 0.0001 \iff |x| < (0.0006)^{1/3} \approx 0.0843.
\]

Hence, the desired range of \( x \) is \([-0.0843 < x < 0.0843]\).
(c) (5 pts) Prove $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ using Euler’s Identity.

Answer: In Euler’s Identity, we substitute $2\theta$ for $\theta$ to get

$$e^{i2\theta} = \cos 2\theta + i\sin 2\theta. \quad (1)$$

On the other hand,

$$e^{i2\theta} = (e^{i\theta})^2 \quad (2)$$
$$= \cos^2 \theta + 2i\cos \theta \sin \theta + i^2 \sin^2 \theta$$
$$= (\cos^2 \theta - \sin^2 \theta) + i(2\sin \theta \cos \theta)$$

Comparing the real part of Equations (1) and (2), we have:

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta.$$

Note also that comparing the imaginary part of Equations (1) and (2), we can also derive:

$$\sin 2\theta = 2\sin \theta \cos \theta.$$