Problem 1 (10 pts) Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{n(x+2)^n}{5^{n-1}}.$$

Solution: Do the ratio test for the absolute convergence. Let $a_n = \dfrac{n(x+2)^n}{5^{n-1}}$. Then,

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)(x+2)^{n+1}}{5^n} \cdot \frac{5^{n-1}}{n(x+2)^n}\right| \\ &= \frac{n+1}{5n}|x+2| \xrightarrow[n\to\infty]{} \frac{|x+2|}{5}. \end{aligned}$$

Hence, the power series converges absolutely if $\dfrac{|x+2|}{5} < 1$. From this, we can see that the *radius of convergence* is $\boxed{R=5}$. Also, the interval of convergence is $-5 < x+2 < 5$, i.e., $-7 < x < 3$.

Let's check the convergence when x is at the boundary points. For $x = -7$, the series becomes:

$$\sum_{n=1}^{\infty} \frac{n(-5)^n}{5^{n-1}} = \sum_{n=1}^{\infty} 5n(-1)^n.$$

Since $\lim_{n\to\infty} 5n(-1)^n \neq 0$, this series does not converge (the nth Term Test for Divergence). So, we cannot include $x = -7$ in the interval of convergence. How about $x = 3$? This leads to

$$\sum_{n=1}^{\infty} \frac{n5^n}{5^{n-1}} = \sum_{n=1}^{\infty} 5n.$$

Clearly this diverges (again via the nth Term Test for Divergence). So, we cannot include $x = 3$ in the interval of convergence either. Hence the *interval of convergence* is: $\boxed{(-7,3)}$ or $\boxed{-7 < x < 3}$.

Score of this page:________________

Problem 2 (10 pts) Consider the function $f(x) = \ln x$.

(a) (5 pts) Approximate $f(x)$ by a Taylor polynomial of degree 2 centered at $x = 2$.

(b) (5 pts) How accurate is this approximation when $1 \le x \le 3$?

Solution to (a): Since $f'(x) = x^{-1}$, $f''(x) = -x^{-2}$, $f'''(x) = 2x^{-3}$, by Taylor's formula, we have

$$\begin{aligned} f(x) &= \ln x \\ &= P_2(x) + R_2(x) \\ &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 + \frac{f'''(c)}{3!}(x-2)^3 \quad 2 < c < x \text{ or } x < c < 2 \\ &= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{3c^3}(x-2)^3. \end{aligned}$$

Hence, we have $\boxed{P_2(x) = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2}$.

Solution to (b): The approximation error is:

$$|R_2(x)| = \frac{1}{3c^3}|x-2|^3 \le \frac{1}{3c^3} < \frac{1}{3},$$

since $|x-2| \le 1$ and $1 < c < 3$. Therefore, the accuracy of approximation by $P_2(x)$ for $1 \le x \le 3$ is bounded by $\boxed{\frac{1}{3} \approx 0.333}$, which is the worst-case scenario.

Problem 3 (10 pts) State and prove the *Cauchy-Schwarz Inequality*. Note that you also need to state and prove the condition for the equality to hold.

Solution: Let $\mathbf{u}$, $\mathbf{v}$ be any two vectors in $\mathbb{R}^n$, $n \in \mathbb{N}$. Then, the Cauchy-Schwarz Inequality is

$\lvert\mathbf{u}\cdot\mathbf{v}\rvert \le \lvert\mathbf{u}\rvert\,\lvert\mathbf{v}\rvert$ where the equality holds if and only if **u** and **v** are parallel to each other.

The proof of this inequality is based on the definition of the dot product. In other words, let θ be the angle between **u** and **v**. Then,

$$\begin{aligned} \lvert\mathbf{u}\cdot\mathbf{v}\rvert &= \big\lvert\lvert\mathbf{u}\rvert\,\lvert\mathbf{v}\rvert\cos\theta\big\rvert \\ &= \lvert\mathbf{u}\rvert\,\lvert\mathbf{v}\rvert\,\lvert\cos\theta\rvert \\ &\le \lvert\mathbf{u}\rvert\,\lvert\mathbf{v}\rvert \quad \text{since clearly } \lvert\cos\theta\rvert \le 1. \end{aligned}$$

From the above derivation, it is also clear that the equality holds if and only if $\cos\theta = 0$ (which includes the case where either **u** or **v** is the zero vector), which is the same as saying that **u** and **v** are parallel to each other.

Score of this page:________________

Problem 4 (10 pts) Let $P(1,4,6)$, $Q(-2,5,-1)$, $R(1,-1,1)$.

(a) (5 pts) Find the area of the triangle $\triangle PQR$.

Hint: Length of the cross product of two vectors is equal to the area of a parallelogram formed by those two vectors.

(b) (5 pts) Find the distance from P to the line QR.

Solution to (a): We have $\overrightarrow{PQ} = \langle -2-1, 5-4, -1-6\rangle = \langle -3, 1, -7\rangle$, and $\overrightarrow{PR} = \langle 1-1, -1-4, 1-6\rangle = \langle 0, -5, -5\rangle$. So,

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = -40\,\mathbf{i} - 15\,\mathbf{j} + 15\,\mathbf{k} = \langle -40, -15, 15\rangle.$$

Hence, the area of $\triangle PQR$ is

$$\frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\left|\langle -40, -15, 15\rangle\right| = \frac{1}{2}\sqrt{2050} = \frac{1}{2}\sqrt{25\cdot 82} = \boxed{\frac{5}{2}\sqrt{82}}.$$

Solution to (b): You can compute the distance d from P to the line QR by the formula based on the cross product, i.e.,

$$d = \frac{\left|\overrightarrow{QP} \times \overrightarrow{QR}\right|}{\left|\overrightarrow{QR}\right|}.$$

Now, notice that

$$\left|\overrightarrow{QP} \times \overrightarrow{QR}\right| = \left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$$

since both amounts to the area of the same parallelogram, which is the twice as the area of the triangle $\triangle PQR$. Since $\overrightarrow{QR} = \langle 3, -6, 2\rangle$, we have

$$d = \frac{5\sqrt{82}}{\sqrt{3^2 + (-6)^2 + 2^2}} = \boxed{\frac{5}{7}\sqrt{82}}.$$

Alternatively, you can reason as follows: the distance d can be computed by the following basic formula

$$\text{Area of } \triangle PQR = \frac{1}{2} d \cdot \left|\overrightarrow{QR}\right|.$$

Since $\overrightarrow{QR} = \langle 3, -6, 2\rangle$, we have

$$d = \frac{2 \times \frac{5}{2}\sqrt{82}}{\sqrt{3^2 + (-6)^2 + 2^2}} = \boxed{\frac{5}{7}\sqrt{82}}.$$

Score of this page:________________

Problem 5 (10 pts) Find the limit if it exists, or show that the limit does not exist.

(a) (5 pts)

$$\lim_{(x,y)\to(0,0)} \frac{y^4}{x^4+3y^4}.$$

Hint: Consider $(x,y)\to(0,0)$ along the line $y=mx$.

(b) (5 pts)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}.$$

Hint: Consider the limit in the polar coordinates (r,θ).

Solution to (a): Consider $(x,y)\to(0,0)$ along the line $y=mx$ for some $m\in\mathbb{R}$. Then,

$$\lim_{(x,y)\to(0,0)} \frac{y^4}{x^4+3y^4} = \lim_{x\to 0} \frac{m^4x^4}{x^4+3m^4x^4} = \lim_{x\to 0} \frac{m^4}{1+3m^4} = \frac{m^4}{1+3m^4}.$$

This value depends on m. Since the limit depends on how the point (x,y) approaches $(0,0)$, $\boxed{\text{the limit does } \textit{not} \text{ exist}}$.

Solution to (b): Consider the polar coordinates: $x=r\cos\theta$, $y=r\sin\theta$, where $r\ge 0$, $0\le\theta<2\pi$.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{\sqrt{r^2(\cos^2\theta+\sin^2\theta)}} = \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{r} = \lim_{r\to 0} r\cos\theta\sin\theta = 0.$$

This is true regardless of the direction of approach θ. Hence, $\boxed{\text{the limit exists and its value is } 0}$.

Score of this page:________________

Problem 6 (10 pts) Verify that the function of two variables

$$u(x,t) = \mathrm{e}^{-\alpha^2 k^2 t} \sin kx,$$

where k is an arbitrary positive integer is a solution of the heat conduction equation over the interval $x \in [0,\pi]$

$$u_t = \alpha^2 u_{xx},$$

with the boundary condition

$$u(0,t) = u(\pi,t) = 0 \quad \text{for all } t \geq 0.$$

Solution: This is a quite straightforward problem.

$$u_t = \frac{\partial u}{\partial t} = -\alpha^2 k^2 \mathrm{e}^{-\alpha^2 k^2 t} \sin kx.$$

$$u_x = \frac{\partial u}{\partial x} = \mathrm{e}^{-\alpha^2 k^2 t} \cdot k \cos kx, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2} = \mathrm{e}^{-\alpha^2 k^2 t} \cdot (-k^2) \sin kx.$$

Hence,

$$\alpha^2 u_{xx} = -\alpha^2 k^2 \mathrm{e}^{-\alpha^2 k^2 t} \sin kx = u_t.$$

Finally, we can easily check that the boundary condition is satisfied since

$$u(0,t) = \mathrm{e}^{-\alpha^2 k^2 t} \sin k0 = 0, \quad u(\pi,t) = \mathrm{e}^{-\alpha^2 k^2 t} \sin k\pi = 0,$$

for any $t \geq 0$.

Score of this page:_______________

Problem 7 (10 pts) Assuming that the equation

$$\sin x + \cos y = \sin x \cos y.$$

defines y as a differentiable function of x, use the Implicit Differentiation Theorem to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution: Let $F(x,y) = \sin x + \cos y - \sin x \cos y = 0$. Then, the Implicit Differentiation Theorem says that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}.$$

Now,

$$F_x(x,y) = \cos x - \cos x \cos y, \quad F_y(x,y) = -\sin y + \sin x \sin y.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\cos x - \cos x \cos y}{-\sin y + \sin x \sin y} = \boxed{\frac{\cos x\,(1-\cos y)}{\sin y\,(1-\sin x)}},$$

as long as $F_y(x,y) = \sin y\,(\sin x - 1) \neq 0$.

Score of this page:______________

Problem 8 (10 pts) Find the direction in which $f(x, y, z) = \ln(xy^2z^3)$

(a) (3 pts) Increases most rapidly at the point $(1, 2, 3)$,

(b) (3 pts) Decreases most rapidly at the point $(1, 2, 3)$.

(c) (4 pts) Does not change (i.e., is flat) at the point $(1, 2, 3)$.

Solution to (a): First of all, note that

$$f(x, y, z) = \ln(xy^2z^3) = \ln x + 2\ln y + 3\ln z.$$

Now, the gradient vector is

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z\rangle = \left\langle \frac{1}{x}, \frac{2}{y}, \frac{3}{z} \right\rangle.$$

So, $\nabla f(1, 2, 3) = \langle 1, 1, 1\rangle$. Now, f increases most rapidly in the direction of ∇f. Hence, this direction is $\boxed{\langle 1, 1, 1\rangle}$.

(It is up to you whether you want to make it to the normal vector of length 1, i.e., $\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$, but it is not necessary in this case.)

Solution to (b): f decreases most rapidly in the direction of $-\nabla f$. Hence, it is $-\langle 1, 1, 1\rangle = \boxed{\langle -1, -1, -1\rangle}$.

Solution to (c): f does not change in the direction orthogonal to ∇f. Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ be this directional vector. Then,

$$\mathbf{u} \cdot \nabla f(1, 2, 3) = 0 \Longleftrightarrow \mathbf{u} = \langle u_1, u_2, u_3\rangle \cdot \langle 1, 1, 1\rangle = 0 \Longleftrightarrow u_1 + u_2 + u_3 = 0.$$

So, $\boxed{\text{along any direction } \mathbf{u} = \langle u_1, u_2, u_3\rangle \text{ satisfying } u_1 + u_2 + u_3 = 0}$, this function f does not change (or is flat) at the point $(1, 2, 3)$.

Score of this page:_______________

Problem 9 (10 pts)

(a) (5 pts) Show the equation of the tangent plane at the point $P_0(x_0, y_0, z_0)$ on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is

$$\frac{x_0\, x}{a^2} + \frac{y_0\, y}{b^2} + \frac{z_0\, z}{c^2} = 1.$$

(b) (5 pts) Compute the normal line to the same ellipsoid at the same point P_0. Furthermore, compute the point where this normal line intersects with the xy-plane. Assume $z_0 \neq 0$.

Solution to (a): Let $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$. The equation of the tangent plane to F at (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Since we have

$$F_x(x, y, z) = \frac{2x}{a^2}, \quad F_x(x, y, z) = \frac{2y}{b^2}, \quad F_x(x, y, z) = \frac{2z}{c^2},$$

the equation of the tangent plane becomes

$$\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) + \frac{2z_0}{c^2}(z - z_0) = 0 \iff \frac{x_0(x - x_0)}{a^2} + \frac{y_0(y - y_0)}{b^2} + \frac{z_0(z - z_0)}{c^2} = 0$$

$$\iff \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1.$$

In the last line, the righthand side becomes 1 since the point $P_0(x_0, y_0, z_0)$ is on this ellipsoid.

Solution to (b): At the point $P_0(x_0, y_0, z_0)$, the gradient vector is $\nabla F(x_0, y_0, z_0) = \langle \frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{2z_0}{c^2} \rangle$. The normal line passes through the point P_0 is parallel to this gradient vector. Hence, the equation of this normal line is

$$\boxed{\begin{cases} x = x_0 + t \cdot \frac{2x_0}{a^2} = x_0 \left(1 + \frac{2}{a^2} t\right) \\ y = y_0 + t \cdot \frac{2y_0}{b^2} = y_0 \left(1 + \frac{2}{b^2} t\right) \\ z = z_0 + t \cdot \frac{2z_0}{c^2} = z_0 \left(1 + \frac{2}{c^2} t\right) \end{cases}},$$

where t is an arbitrary real value.

As for the intersection of this normal line and the xy-plane, because xy-plane is the same as the equation $z = 0$, we seek the value of t that makes $z = 0$. This means that $t = -\frac{c^2}{2}$ since $z_0 \neq 0$. Hence, we insert this t into the above equation to get the coordinate of the intersection $\boxed{(x, y, z) = \left(x_0 \left(1 - \frac{c^2}{a^2}\right), y_0 \left(1 - \frac{c^2}{b^2}\right), 0\right)}$.

Score of this page:________________

Problem 10 (10 pts) Find the linearization of $f(x,y,z)=\sqrt{x^2+y^2+z^2}$ at the point $(3,2,6)$. Then use it to approximate $f(3.1,1.9,6.2)$ by the three decimal point number. Note that the true value of $f(3.1,1.9,6.2)$ is 7.188 in the three decimal point number.

Solution: The linearization of $f(x,y,z)$ at the point (x_0,y_0,z_0) is

$$L(x,y,z)=f(x_0,y_0,z_0)+f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0).$$

We have

$$f_x(x,y,z)=\frac{1}{2}\cdot(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 2x=x(x^2+y^2+z^2)^{-\frac{1}{2}}.$$

Since $f(x,y,z)$ is symmetric with respect to x, y, z, we immediately have

$$f_y(x,y,z)=y(x^2+y^2+z^2)^{-\frac{1}{2}},\quad f_z(x,y,z)=z(x^2+y^2+z^2)^{-\frac{1}{2}}.$$

Now, at the point $(3,2,6)$, we have $f(3,2,6)=\sqrt{3^2+2^2+6^2}=\sqrt{49}=7$. Hence,

$$\boxed{L(x,y,z)=7+\frac{3}{7}(x-3)+\frac{2}{7}(y-2)+\frac{6}{7}(z-6)}.$$

Let's now use this to approximate $f(3.1,1.9,6.2)$.

$$L(3.1,1.9,6.2)=7+\frac{3}{7}(3.1-3)+\frac{2}{7}(1.9-2)+\frac{6}{7}(6.2-6)=7+\frac{1.3}{7}\approx\boxed{7.186}.$$

On the other hand, the true value is $f(3.1,1.9,6.2)=7.188$ as the hint suggests. So, the approximation is quite close to the true value and the error is $\boxed{|f(3.1,1.9,6.2)-L(3.1,1.9,6.2)|=0.002}$. Observe that this linear approximation is quite good!

Score of this page:________________

Problem 11 (10 pts) Find the absolute maximum and minimum values of

$$f(x,y) = x^2 - xy + y^2 + 1,$$

on the closed triangular domain bounded by the lines $x = 0,\ y = 2,\ y = 2x$, i.e.,

$$\Omega = \{(x,y)\,|\,x \geq 0, y \leq 2, y \geq 2x\}.$$

Solution: The domain Ω is the shaded triangle in the figure.

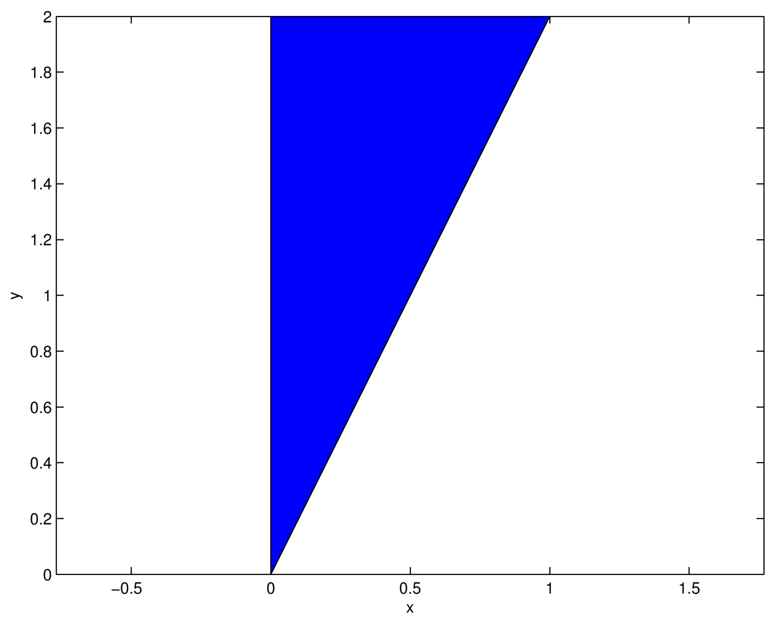

Figure 1: A triangular domain Ω.

Step 1: Compute all the critical points in Ω.

$$f_x = 2x - y = 0,\ f_y = -x + 2y = 0 \Longleftrightarrow y = 2x,\ x = 2y \Longleftrightarrow x = 4x \Longleftrightarrow x = 0.$$

Hence, $x = y = 0$ is the only critical point in Ω. Now we need to check whether this point is local min., local max., or a saddle point. To do so, we compute

$$f_{xx}(x,y) = 2,\ f_{yy}(x,y) = 2,\ f_{xy}(x,y) = -1.$$

From these, we do the *Second Derivative Test*:

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - \left(f_{xy}(0,0)\right)^2 = 2 \cdot 2 - (-1)^2 = 3 > 0.$$

Since $D(0,0) > 0$ and $f_{xx}(0,0) = 2 > 0$, the value $f(0,0) = 1$ is the *local minimum*.

Score of this page:________________

Step 2 : Find the extreme values at the boundary of Ω.

On $x = 0$: $f(0, y) = y^2 + 1$, $0 \le y \le 2$. So, on this boundary, the minimum is $f(0,0) = 1$ while the maximum is $f(0,2) = 5$.

On $y = 2$: $f(x,2) = x^2 - 2x + 5 = (x-1)^2 + 4$, $0 \le x \le 1$. So, on this boundary, the minimum is $f(1,2) = 4$ while the maximum is $f(0,2) = 5$.

On $y = 2x$: $f(x,2x) = 3x^2 + 1$, $0 \le x \le 1$. So, on this boundary, the minimum is $f(0,0) = 1$ while the maximum is $f(1,2) = 4$.

Step 3: Find the largest and the smallest values from the results of Steps 1 and 2.

Combining all the results we got so far, we can easily conclude that
the absolute maximum is $f(0,2) = 5$ while the absolute minimum is $f(0,0) = 1$.

Problem 12 (10 pts) Use Lagrange multipliers to find the maximum and minimum values of the function

$$f(x,y) = \mathrm{e}^{xy} \quad \text{subject to} \quad x^2 + y^2 = 1.$$

Note that we only consider the real values for x and y, not the complex values.

Solution: Let $g(x,y) = x^2 + y^2 - 1 = 0$ be the constraint. Let's find the points (x,y) and the parameter λ satisfying

$$\nabla f = \lambda \nabla g, \quad g(x,y) = 0. \tag{1}$$

We have

$$\nabla f = \langle f_x, f_y \rangle = \langle y\mathrm{e}^{xy}, x\mathrm{e}^{xy} \rangle, \quad \nabla g = \langle g_x, g_y \rangle = \langle 2x, 2y \rangle.$$

Hence, Eq. (1) leads to

$$\begin{aligned} y\mathrm{e}^{xy} &= 2\lambda x && (2) \\ x\mathrm{e}^{xy} &= 2\lambda y && (3) \\ x^2 + y^2 &= 1. && (4) \end{aligned}$$

We can see that $\lambda \neq 0$ because if $\lambda = 0$, then $x = y = 0$ from Eqn's (2) and (3), which contradicts Eq. (4). Similarly, $x \neq 0$ as well as $y \neq 0$. Hence, we can divide Eq. (2) by Eq. (3) to get

$$\frac{y\mathrm{e}^{xy}}{x\mathrm{e}^{xy}} = \frac{2\lambda x}{2\lambda y} \Longleftrightarrow \frac{y}{x} = \frac{x}{y} \Longleftrightarrow x^2 = y^2.$$

This implies $x = y$ or $x = -y$. Insert these into Eq. (4) gives us $2x^2 = 1$ or $x = \pm\frac{1}{\sqrt{2}}$. So, we have $(x,y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, or $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. From Eq. (2), the corresponding λ's can be easily calculated as $\lambda = \frac{\sqrt{\mathrm{e}}}{2}$, $-\frac{1}{2\sqrt{\mathrm{e}}}$, $-\frac{1}{2\sqrt{\mathrm{e}}}$, or $\frac{\sqrt{\mathrm{e}}}{2}$, respectively. Hence,

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \sqrt{\mathrm{e}},$$

$$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{\mathrm{e}}}.$$

Therefore, the maxima are

$$\boxed{f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \sqrt{\mathrm{e}}},$$

and the minima are

$$\boxed{f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{\mathrm{e}}}}.$$

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