Math 22B: Additional Exercises

Sec. 7.6: 8, 13, 28
Sec. 7.7: 1, 10, 13
Sec. 7.8: 5, 8, 17, 19

1 Sec. 7.6

7.6.8) Given the system of ODE's

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

where

$$\mathbf{A} = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix}$$

then as discussed before, if we assume the solutions are of the form

$$\mathbf{x} = \boldsymbol{\xi} e^{rt}$$

then r and $\boldsymbol{\xi}$ are the eigenvalues and eigenvectors of the matrix $\mathbf{A}$. To find the eigenvalues we set

$$\det(\mathbf{A} - r\mathbf{I}) = 0$$

or

$$\begin{vmatrix} -3-r & 0 & 2 \\ 1 & -1-r & 0 \\ -2 & -1 & 0-r \end{vmatrix} = 0$$

$$(-3-r)[(-1-r)(-r) - 0] - 0 + 2[(1)(-1) - (-2)(-1-r)] = 0$$

$$r^3 + 4r^2 + 7r + 6 = 0.$$

Now, we note by trial and error that -2 is a root of the above polynomial. So, the polynominal can be rewritten as

$$(r+2)(r^2 + 2r + 3) = 0.$$

Now

$$r^2 + 2r + 3 = 0$$

gives us

$$r = \frac{-2 \pm \sqrt{4 - 12}}{2}$$

or

$$r = -1 \pm i\sqrt{2}.$$

Now, to find the eigenvectors, $\boldsymbol{\xi}$, we note that for $r = -2$,

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$$

becomes
$$\begin{pmatrix} -3-(-2) & 0 & 2 \\ 1 & -1-(-2) & 0 \\ -2 & -1 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \mathbf{0}$$
or
$$\begin{cases} -\xi_1 + 2\xi_3 = 0 \\ \xi_1 + \xi_2 = 0 \end{cases}.$$
So, if we let $\xi_3 = 1$, then $xi_1 = 2$ and $xi_2 = -2$. Hence, the eigenvector associated with $r = -2$ is
$$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$
For $r = -1 + i\sqrt{2}$, we have that
$$\begin{pmatrix} -3-(-1+i\sqrt{2}) & 0 & 2 \\ 1 & -1-(-1+i\sqrt{2}) & 0 \\ -2 & -1 & -1+i\sqrt{2} \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \mathbf{0}$$
or
$$\begin{cases} (-2-i\sqrt{2})\xi_1 + 2\xi_3 = 0 \\ \xi_1 - i\sqrt{2}\xi_2 = 0 \end{cases}.$$
Now, let $\xi_2 = 1$. Then $\xi_1 = i\sqrt{2}$ and
$$\xi_3 = \frac{(2+i\sqrt{2})i\sqrt{2}}{2} = -1 + i\sqrt{2}.$$
So, the solution associated with the complex eigenvalue $r = -1 + i\sqrt{2}$ is
$$\begin{aligned}
\mathbf{x} &= \begin{pmatrix} i\sqrt{2} \\ 1 \\ -1+i\sqrt{2} \end{pmatrix} e^{(-1+i\sqrt{2})t} \\
&= e^{-t}\begin{pmatrix} i\sqrt{2} \\ 1 \\ -1+i\sqrt{2} \end{pmatrix}(\cos(\sqrt{2}t) + i\sin(\sqrt{2}t)) \\
&= e^{-t}\begin{pmatrix} -\sqrt{2}\sin(\sqrt{2}t) + i\sqrt{2}\cos(\sqrt{2}t) \\ \cos(\sqrt{2}t) + i\sin(\sqrt{2}t) \\ -\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) + i(\sqrt{2}\cos(\sqrt{2}t) - \sin(\sqrt{2}t)) \end{pmatrix} \\
&= e^{-t}\begin{pmatrix} -\sqrt{2}\sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) \\ -\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \end{pmatrix} + ie^{-t}\begin{pmatrix} \sqrt{2}\cos(\sqrt{2}t) \\ \sin(\sqrt{2}t) \\ \sqrt{2}\cos(\sqrt{2}t) - \sin(\sqrt{2}t) \end{pmatrix}
\end{aligned}$$
So, the general solution is
$$\mathbf{x} = c_1e^{-2t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + c_2e^{-t}\begin{pmatrix} -\sqrt{2}\sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) \\ -\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \end{pmatrix} + c_3e^{-t}\begin{pmatrix} \sqrt{2}\cos(\sqrt{2}t) \\ \sin(\sqrt{2}t) \\ \sqrt{2}\cos(\sqrt{2}t) - \sin(\sqrt{2}t) \end{pmatrix}.$$

7.6.13) Given

$$\mathbf{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \mathbf{x},$$

then as above, we need to find the eigenvalues, r, as follows:

$$\begin{vmatrix} \alpha - r & 1 \\ -1 & \alpha - r \end{vmatrix} = 0$$

or

$$\begin{aligned} (\alpha - r)(\alpha - r) + 1 = 0 \\ (\alpha - r)^2 = -1 \\ \alpha - r = \pm i \\ r = \alpha \pm i. \end{aligned}$$

For, $r = \alpha + i$, we find the eigenvector, $\boldsymbol{\xi}$, from

$$\begin{pmatrix} \alpha - (\alpha + i) & 1 \\ -1 & \alpha - (\alpha + i) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$$

or

$$-i\xi_1 + \xi_2 = 0.$$

So, if we let $\xi_1 = 1$, then $\xi_2 = i$. Hence

$$\boldsymbol{\xi} = \begin{pmatrix} 1 \\ i \end{pmatrix},$$

and so the solution for the case $r = \alpha + i$ is

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} 1 \\ i \end{pmatrix} e^{\alpha + i} \\ &= e^{\alpha t} \begin{pmatrix} 1 \\ i \end{pmatrix} [\cos t + i \sin t] \\ &= e^{\alpha t} \begin{pmatrix} \cos t + i \sin t \\ i[\cos t + i \sin t] \end{pmatrix} \\ &= e^{\alpha t} \begin{pmatrix} \cos t + i \sin t \\ -\sin t + i \cos t \end{pmatrix} \\ &= e^{\alpha t} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i e^{\alpha t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \end{aligned}$$

So, the general solution is

$$\mathbf{x} = c_1 e^{\alpha t} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^{\alpha t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}. \tag{1}$$

Note that for any real α, the eigenvalues will always be complex. This means that the solutions will always be some combination of sines and cosines. However, if $\alpha > 0$, then $e^{\alpha t} \to \infty$ as $t \to \infty$ (and $e^{\alpha t} \to 0$ as $t \to -\infty$) meaning that the solutions would blow up in time and if $\alpha < 0$, $e^{\alpha t} \to 0$ and so the solutions would decay in time. So $\alpha = 0$ is the critical value. Fig. 1 is a phase portrait.

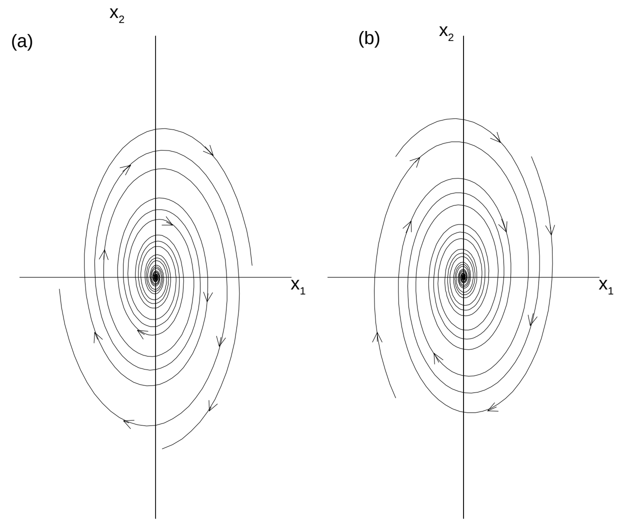

Figure 1: (a) is a phase portrait for $\alpha > 0$ and (b) is a phase portrait for $\alpha < 0$.

7.6.28) Given

$$mu'' + ku = 0,$$

if we let

$$x_1 = u \text{ and } x_2 = u'$$

then

$$x_1' = u' = x_2 \text{ and } x_2' = u''. \tag{2}$$

So

$$mu'' + ku = 0$$

can be rewritten as

$$mx_2' + kx_1 = 0$$

or

$$x_2' = -\frac{k}{m}x_1.$$

Now, using the above equation and Eqn. (2), we get our system of equations

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{k}{m}x_1 \end{cases}$$

or

$$\begin{cases} x_1' = 0x_1 + x_2 \\ x_2' = -\frac{k}{m}x_1 + 0x_2 \end{cases}$$

and so in matrix form

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(b) To find the eigenvalues, r, we have

$$\begin{vmatrix} 0-r & 1 \\ -\frac{k}{m} & 0-r \end{vmatrix} = 0$$

or

$$r^2 + \frac{k}{m} = 0$$

$$r = \pm i\sqrt{\frac{k}{m}}.$$

Now, for $r = i\sqrt{\frac{k}{m}}$, we have that

$$\begin{pmatrix} 0-i\sqrt{\frac{k}{m}} & 1 \\ -\frac{k}{m} & 0-i\sqrt{\frac{k}{m}} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$$

or

$$-i\sqrt{\frac{k}{m}}\xi_1 + \xi_2 = 0.$$

So, if we let $\xi_1 = 1$, then $\xi_2 = i\sqrt{\frac{k}{m}}$. Hence our eigenvector is

$$\begin{pmatrix} 1 \\ i\sqrt{\frac{k}{m}} \end{pmatrix},$$

which gives us as a solution

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} 1 \\ i\sqrt{\frac{k}{m}} \end{pmatrix} e^{i\sqrt{\frac{k}{m}}t} \\ &= \begin{pmatrix} \cos(\sqrt{\frac{k}{m}}t) + i\sin(\sqrt{\frac{k}{m}}t) \\ -\sin(\sqrt{\frac{k}{m}}t) + i\cos(\sqrt{\frac{k}{m}}t) \end{pmatrix}. \end{aligned}$$

So, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} \cos(\sqrt{\frac{k}{m}}t) \\ -\sin(\sqrt{\frac{k}{m}}t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(\sqrt{\frac{k}{m}}t) \\ \cos(\sqrt{\frac{k}{m}}t) \end{pmatrix}.$$

(c) Note that since the general solution does not have an exponential in front of the sines and cosines, the solutions are periodic (see Fig. 2.)

(d) From the general solution above, it is clear that

$$|r| = \sqrt{\frac{k}{m}}$$

is the natural frequency of the system.

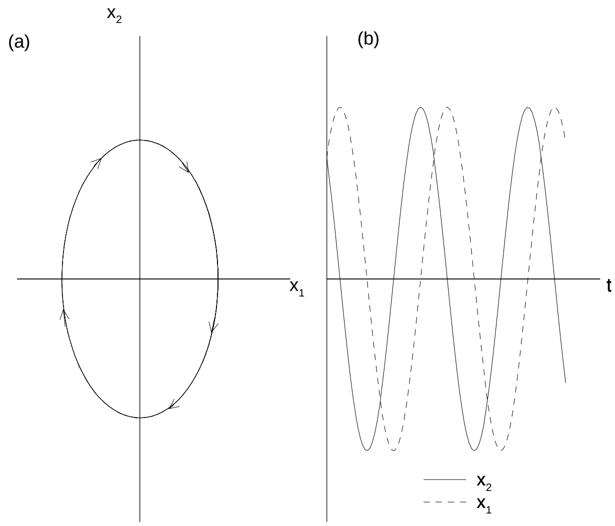

Figure 2: (a) is a phase portrait for $\frac{k}{m}$, c_1 and c_2 set to various values and (b) is a plot of x_1 and x_2 verses time for $\frac{k}{m} = 4$.

2 Sec. 7.7

7.7.1) Given

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$$

the characteristic equation is

$$\begin{vmatrix} 3-r & -2 \\ 2 & -2-r \end{vmatrix} = 0$$

or

$$(3-r)(-2-r)+4=0$$
$$r^2 - r - 2 = 0$$
$$(r-2)(r+1) = 0$$
$$r = 2 \text{ and } r = -1.$$

For $r = 2$ we have

$$\begin{pmatrix} 3-2 & -2 \\ 2 & -2-2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \mathbf{0}$$

or

$$\xi_1 - 2\xi_2 = 0.$$

So, the first eigenvector is

$$\boldsymbol{\xi}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and the first solution is

$$\mathbf{x}^{(1)} = c_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

For, $r = -1$, we have

$$4\xi_1 - 2\xi_2 = 0$$

and so

$$\boldsymbol{\xi}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Hence, the second solution is

$$\mathbf{x}^{(2)} = c_2 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

So, the general solution is

$$\mathbf{x} = c_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and the fundamental matrix is

$$\Psi = \begin{pmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix}.$$

To find Φ, we first need to find c_1 and c_2 such that the general solution satistfies

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

or

$$c_1 e^{2(0)} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-(0)} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This gives us a system of equations

$$\begin{cases} 2c_1 + c_2 = 1 \\ c_1 + 2c_2 = 0 \end{cases}$$

or

$$c_1 = \frac{2}{3} \text{ and } c_2 = -\frac{1}{3}.$$

So

$$\mathbf{x} = \frac{2}{3} e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{1}{3} e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

satisfies the first initial condition. Now the second initial condition that must be satisfied is

$$\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which gives us

$$\begin{cases} 2c_1 + c_2 = 0 \\ c_1 + 2c_2 = 1 \end{cases}$$

or

$$c_1 = -\frac{1}{3} \text{ and } c_2 = \frac{2}{3}.$$

So

$$\mathbf{x} = -\frac{1}{3}e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{2}{3}e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

satisfies the second initial condition. Hence

$$\Phi = \begin{pmatrix} 2\left(\frac{2}{3}\right)e^{2t} - \frac{1}{3}e^{-t} & -2\left(\frac{1}{3}\right)e^{2t} + \frac{2}{3}e^{-t} \\ \frac{2}{3}e^{2t} - 2\left(\frac{1}{3}\right)e^{-t} & -\frac{1}{3}e^{2t} + 2\left(\frac{2}{3}\right)e^{-t} \end{pmatrix}.$$

7.7.10) Given

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}\mathbf{x}$$

the corresponding characteristic equation is

$$\begin{vmatrix} 1-r & -1 & 4 \\ 3 & 2-r & -1 \\ 2 & 1 & -1-r \end{vmatrix} = 0$$

or

$$(1-r)[(2-r)(-1-r)+1] + [3(-1-r)+2] + 4[3-2(2-r)] = 0$$

$$-r^3 + 2r^2 + 5r - 6 = 0$$

$$(r+2)(-r^2 + 4t - 3) = 0$$

$$(r+2)(-r+3)(r-1) = 0$$

$$r = -2, \quad r = 3 \text{ or } r = 1.$$

For $r = -2$ we have

$$\begin{pmatrix} 1-(-2) & -1 & 4 \\ 3 & 2-(-2) & -1 \\ 2 & 1 & -1-(-2) \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \mathbf{0}$$

or

$$\begin{cases} 3\xi_1 - \xi_2 + 4\xi_3 = 0 \\ 3\xi_1 + 4\xi_2 - \xi_3 = 0 \end{cases}$$

which gives us

$$\xi_2 = \xi_3$$

and

$$\xi_1 = \frac{1}{3}\xi_2 - \frac{4}{3}\xi_3 = -\xi_2$$

So, the eigenvector associated with $r = -2$ is

$$\boldsymbol{\xi}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

For $r = 3$, we have

$$\begin{cases} -2\xi_1 - \xi_2 + 4\xi_3 = 0 \\ 3\xi_1 - \xi_2 - \xi_3 = 0. \end{cases}$$

So

$$\xi_1 = \xi_3$$

and

$$\xi_2 = -2\xi_1 + 4\xi_3 = 2\xi_1.$$

Hence, the eigenvector associated with $r = 3$ is

$$\boldsymbol{\xi}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

For $r = 1$, we have

$$\begin{cases} -\xi_2 + 4\xi_3 = 0 \\ 3\xi_1 + \xi_2 - \xi_3 = 0 \end{cases}$$

which gives us

$$\xi_2 = 4\xi_3$$

and

$$\xi_1 = -\frac{1}{3}\xi_2 + \frac{1}{3}\xi_3 = -\xi_3.$$

So

$$\boldsymbol{\xi}_3 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}.$$

Therefore, our general solution is

$$\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_3 e^{t} \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}.$$

So,

$$\Psi = \begin{pmatrix} -e^{-2t} & e^{3t} & -e^{t} \\ e^{-2t} & 2e^{3t} & 4e^{t} \\ e^{-2t} & e^{3t} & e^{t} \end{pmatrix}.$$

To find $\Phi(t)$, we use the formula $\Phi(t) = \Psi(t)\Psi(t_0)^{-1}$, and in this case $t_0 = 0$.

$$\Psi(0) = \begin{pmatrix} -1 & 1 & -1 \\ -1 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix}.$$

So, we get:

$$\Psi(0)^{-1} = \begin{pmatrix} -1/3 & -1/3 & 1 \\ 1/2 & 0 & 1/2 \\ -1/6 & 1/3 & -1/2 \end{pmatrix}.$$

Finally we get:

$$\begin{aligned} \Phi(t) = \Psi(t)\Psi(0)^{-1} &= \begin{pmatrix} -e^{-2t} & e^{3t} & -e^{t} \\ e^{-2t} & 2e^{3t} & 4e^{t} \\ e^{-2t} & e^{3t} & e^{t} \end{pmatrix} \begin{pmatrix} -1/3 & -1/3 & 1 \\ 1/2 & 0 & 1/2 \\ -1/6 & 1/3 & -1/2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3}e^{-2t} + \frac{1}{2}e^{3t} + \frac{1}{6}e^{t} & \frac{1}{3}e^{-2t} - \frac{1}{3}e^{t} & -e^{-2t} + \frac{1}{2}e^{3t} + \frac{1}{2}e^{t} \\ -\frac{1}{3}e^{-2t} + e^{3t} - \frac{2}{3}e^{t} & -\frac{1}{3}e^{-2t} + \frac{4}{3}e^{t} & e^{-2t} + e^{3t} - 2e^{t} \\ -\frac{1}{3}e^{-2t} + \frac{1}{2}e^{3t} - \frac{1}{6}e^{t} & -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t} & e^{-2t} + \frac{1}{2}e^{3t} - \frac{1}{2}e^{t} \end{pmatrix}. \end{aligned}$$

7.7.13) We already did this in class. But let's do this again here. Let $\Psi(t)$ be any fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$. Then, the general solution can be written as $\mathbf{x}(t) = \Psi(t)\mathbf{c}$, where $\mathbf{c}$ is a vector of length n consisting of arbitrary constants. Now, let the initial condition be $\mathbf{x}(t_0) = \mathbf{x}^0 = \Psi(t_0)\mathbf{c}$. Hence, $\mathbf{c} = \Psi(t_0)^{-1}\mathbf{x}^0$, i.e.,

$$\mathbf{x}(t) = \Psi(t)\Psi(t_0)^{-1}\mathbf{x}^0.$$

Therefore, $\Psi(t)\Psi(t_0)^{-1}$ is also a fundamental matrix for this system and satisfies $\Psi(t_0)\Psi(t_0)^{-1} = I$. In other words, $\Phi(t) = \Psi(t)\Psi(t_0)^{-1}$.

3 Sec. 7.8

7.8.5)

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{x}$$

Let us first compute the eigenvalues.

$$\begin{vmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ 0 & -1 & 1-r \end{vmatrix} = (1-r)^3 - 2 - 2(1-r) - (1-r) = 0.$$

This yields

$$0 = r^3 - 3r^2 + 4 = (r+1)(r-2)^2$$

Hence, the eigenvalues of this matrix, say A, are $r = -1$ and $r = 2$ (double root). For $r = -1$, we can compute the eigenvector $\boldsymbol{\xi}$ as follows:

$$(A+I)\boldsymbol{\xi} = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

From this, we have:

$$\begin{aligned} 2\xi_1 + \xi_2 + \xi_3 &= 0, \\ -\xi_2 + 2\xi_3 &= 0. \end{aligned}$$

So, we have: $\xi_2 = 2\xi_3$, and $\xi_1 = -(\xi_2 + \xi_3)/2$. By setting $\xi_3 = 2$, we get $\boldsymbol{\xi} = (-3, 4, 2)^T$. Thus,

$$\mathbf{x}^{(1)} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} e^{-t}.$$

Now let us compute the eigenvector $\boldsymbol{\eta}$ corresponding $r = 2$.

$$(A-2I)\boldsymbol{\xi} = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

From this, we have:

$$\begin{aligned} -\eta_1 + \eta_2 + \eta_3 &= 0, \\ -\eta_2 - \eta_3 &= 0. \end{aligned}$$

Thus, adding these two, we get $\eta_1 = 0$, $\eta_2 = -\eta_3$. By setting $\eta_2 = 1$, we get $\eta_3 = -1$, and $\boldsymbol{\eta} = (0, 1, -1)^T$. Hence

$$\mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}.$$

Now, it is easy to see that $\boldsymbol{\eta}$ is the only linearly independent eigenvector for $r = 2$ since $\eta_1 = 0$ is specified and $\eta_2 = -\eta_3$ must be satisfied. Thus, we need to compute $\boldsymbol{\zeta}$ satisfying $(A - 2I)\boldsymbol{\zeta} = \boldsymbol{\eta}$. This is written as

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

From this, we get:

$$\begin{aligned} -\zeta_1 + \zeta_2 + \zeta_3 &= 0, \\ -\zeta_2 - \zeta_3 &= -1. \end{aligned}$$

By adding these two equations, we get $\zeta_1 = 1$. Also we get $\zeta_2 + \zeta_3 = 1$. Setting $\zeta_2 = 0$, we get $\zeta_3 = 1$. So, $\boldsymbol{\zeta} = (1, 0, 1)^T + k\boldsymbol{\eta}$, where k is an arbitrary constant. Note that $(A - 2I)\boldsymbol{\zeta} = \mathbf{0}$ since $(A - 2I)\boldsymbol{\eta} = \mathbf{0}$. Thus, we can set $k = 0$. Hence, we have

$$\mathbf{x}^{(3)} = \boldsymbol{\eta} te^{2t} + \boldsymbol{\zeta} e^{2t} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}$$

Finally, we can write the general solution as:

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = c_1 \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} + c_3 \left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} \right].$$

7.8.8) Given

$$\mathbf{x}' = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x},$$

we find the eigenvalues of the matrix in our usual way as follows:

$$\begin{vmatrix} -\frac{5}{2} - r & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} - r \end{vmatrix} = 0$$

or

$$\begin{aligned} \left(-\frac{5}{2} - r\right)\left(\frac{1}{2} - r\right) + \left(\frac{3}{2}\right)^2 &= 0 \\ r^2 + 2r + 1 &= 0 \\ (r+1)^2 &= 0 \\ r &= -1. \end{aligned}$$

Now, to find the eigenvector, $\boldsymbol{\xi}$, associated with the eigenvalue $r = -1$, we look at the system

$$\begin{pmatrix} -\frac{5}{2} - (-1) & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} - (-1) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \mathbf{0}$$

or

$$\begin{aligned} -\frac{3}{2}\xi_1 + \frac{3}{2}\xi_2 &= 0 \\ \xi_1 &= \xi_2. \end{aligned}$$

So, the eigenvector is

$$\boldsymbol{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence, one of the solutions is

$$\mathbf{x}^{(1)} = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since we have only one eigenvalue and more importantly, only one eigenvector associated with that eigenvalue, we assume that the second solution has the form

$$\mathbf{x}^{(2)} = \boldsymbol{\xi} t e^{-t} + \boldsymbol{\eta} e^{-t}.$$

Now, to find $\boldsymbol{\eta}$, we need to solve the system of equations

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$$

or

$$\begin{pmatrix} -\frac{5}{2} - (-1) & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} - (-1) \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

or

$$\begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This gives us

$$-\frac{3}{2}\eta_1 + \frac{3}{2}\eta_2 = 1$$

$$\eta_2 = \frac{2}{3} + \eta_1.$$

So, if we let $\eta_1 = k$, for some constant k, then $\eta_2 = \frac{2}{3} + k$. In vector form, this is

$$\boldsymbol{\eta} = \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for some constant k. Hence, our second equation is

$$\begin{aligned} \mathbf{x}^{(2)} &= \boldsymbol{\xi} t e^{-t} + \boldsymbol{\eta} e^{-t} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix} e^{-t} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}. \end{aligned}$$

Note that the last term is just a multiple of the first solution. So, we can ignore it in the second solution, i. e.

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix} e^{-t}.$$

So, our general solution is

$$\mathbf{x} = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 t e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix}.$$

Now, the initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

gives us

$$c_1 e^0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 t e^0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^0 \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

or

$$\begin{cases} c_1 = 3 \\ c_1 + \frac{2}{3}c_2 = -1 \end{cases}.$$

So

$$c_1 = 3 \text{ and } c_2 = -6.$$

Therefore, the solution to our initial value problem is

$$\begin{aligned} \mathbf{x} &= 3e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 6te^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 6e^{-t}\begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix} \\ &= e^{-t}\begin{pmatrix} 3 \\ 3 - 6\left(\frac{2}{3}\right) \end{pmatrix} - 6te^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= e^{-t}\begin{pmatrix} 3 \\ -1 \end{pmatrix} - 6te^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

7.8.17) Given

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}\mathbf{x}.$$

(a) The characteristic equation to find the eigenvalues, r, is

$$\begin{vmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ -3 & 2 & 4-r \end{vmatrix} = 0$$

or

$$(1-r)[(1-r)(4-r) - (-2)] - [2(4-r) - 3] + [4 - (-3)(1-r)] = 0$$

$$-r^3 + 6r^2 - 12r + 8 = 0$$

$$(r-2)(-r^2 + 4r - 4) = 0.$$

Now

$$-r^2 + 4r - 4 = 0$$

gives us

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$r = 2.$$

So $r = 2$ is a triple root of the characteristic equation.

Now to find the eigenvector, $\boldsymbol{\xi}$, we have

$$\begin{pmatrix} 1-2 & 1 & 1 \\ 2 & 1-2 & -1 \\ -3 & 2 & 4-2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \mathbf{0}$$

or

$$\begin{cases} -\xi_1 + \xi_2 + \xi_3 = 0 \\ 2\xi_1 - \xi_2 - \xi_3 = 0 \end{cases}.$$

Note that the third equation is just a linear combination of the first two and so does not contribute anything. This, of course, will always be the case because of the way we chose r. Now, if we add the two equations, we get

$$\xi_1 = 0$$

which gives us

$$\xi_2 + \xi_3 = 0$$

or

$$\xi_2 = -\xi_3.$$

So, if we let $\xi_3 = -1$, then our eigenvector is

$$\boldsymbol{\xi} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Let us denote this vector as $\boldsymbol{\xi}^{(1)}$ for future reference.

(b) Using the information in (a), our first solution is

$$\mathbf{x}^{(1)} = \boldsymbol{\xi}^{(1)} e^{2t} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} .e^{2t}$$

(c) To find our second solution, we assume it has the form

$$\mathbf{x}^{(2)} = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}.$$

So, to find $\boldsymbol{\eta}$, we look at the equations

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$$

or

$$\begin{pmatrix} 1-2 & 1 & 1 \\ 2 & 1-2 & -1 \\ -3 & 2 & 4-2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

This gives us

$$\begin{cases} -\eta_1 + \eta_2 + \eta_3 = 0 \\ 2\eta_1 - \eta_2 - \eta_3 = 1 \end{cases}$$

Adding the two equations gives us

$$\eta_1 = 1.$$

So

$$\eta_2 = 1 - \eta_3.$$

Hence, we get

$$\boldsymbol{\eta} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + k \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

This means that our second solution is

$$\begin{aligned} \mathbf{x}^{(2)} &= \boldsymbol{\xi} te^{2t} + \boldsymbol{\eta} e^{2t} \\ &= \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + k \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} \end{aligned}$$

and since the last term is just a multiple of the first solution, $\mathbf{x}^{(1)}$, we can drop that term and get

$$\mathbf{x}^{(2)} = \left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} \right].$$

(d) Now, to find the third solution, we assume it to be of the form

$$\mathbf{x}^{(3)} = \boldsymbol{\xi} \frac{t^2}{2!} e^{2t} + \boldsymbol{\eta} te^{2t} + \boldsymbol{\zeta} e^{2t}.$$

This means that $\boldsymbol{\zeta}$ must satisfy

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\zeta} = \boldsymbol{\eta}$$

or

$$\begin{pmatrix} 1-2 & 1 & 1 \\ 2 & 1-2 & -1 \\ -3 & 2 & 4-2 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1+k \\ -k \end{pmatrix}.$$

The above gives us

$$\begin{cases} -\zeta_1 + \zeta_2 + \zeta_3 = 1 \\ 2\zeta_1 - \zeta_2 - \zeta_3 = 1 + k \end{cases}$$

and adding the two equations gives us

$$\zeta_1 = 2 + k.$$

So

$$\zeta_3 = 1 + \zeta_1 - \zeta_2 = 3 + k - \zeta_2.$$

Hence by setting $\zeta_2 = \ell$, where ℓ is arbitrary, we have

$$\boldsymbol{\zeta} = \begin{pmatrix} 2+k \\ \ell \\ 3+k-\ell \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + k \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \ell \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Again by ignoring the second and third terms (i.e., setting $k = \ell = 0$), the third solution becomes

$$\begin{aligned}\mathbf{x}^{(3)} &= \boldsymbol{\xi}\frac{t^2}{2!}e^{2t} + \boldsymbol{\eta} te^{2t} + \boldsymbol{\zeta} e^{2t} \\ &= \frac{t^2}{2}e^{2t}\begin{pmatrix}0\\1\\-1\end{pmatrix} + te^{2t}\begin{pmatrix}1\\1\\0\end{pmatrix} + e^{2t}\begin{pmatrix}2\\0\\3\end{pmatrix}\end{aligned}$$

(e) Since $\Psi(t) = \left[\mathbf{x}^{(1)} \,|\, \mathbf{x}^{(2)} \,|\, \mathbf{x}^{(3)}\right]$, we have:

$$\Psi(t) = e^{2t}\begin{pmatrix}0 & 1 & t+2\\ 1 & t+1 & \frac{t^2}{2}+t\\ -1 & -t & -\frac{t^2}{2}+3\end{pmatrix}.$$

(f) Since $T = [\boldsymbol{\xi} \,|\, \boldsymbol{\eta} \,|\, \boldsymbol{\zeta}]$,

$$T = \begin{pmatrix}0 & 1 & 2\\ 1 & 1 & 0\\ -1 & 0 & 3\end{pmatrix}.$$

We can compute T^{-1} and get:

$$T^{-1} = \begin{pmatrix}-3 & 3 & 2\\ 3 & -2 & -2\\ -1 & 1 & 1\end{pmatrix}.$$

The matrix J is written as

$$J = \begin{pmatrix}2 & 1 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{pmatrix}.$$

You can verify that $J = T^{-1}AT$ and $A = TJT^{-1}$. This sandwich is called the *Jordan form* of A. As we can see from this example, given any matrix A of $n \times n$, we may not be able to compute the eigenvalue decomposition, but we can always write A by its Jordan form. If A has all distinct n eigenvalues or A has some repeated eigenvalues, but the algebraic multiplicity of each distinct eigenvalue is the same as its geometric multiplicity, then the Jordan form reduces to the eigenvalue decomposition.

7.7.19) Given

$$\mathbf{A} = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$$

(a)

$$\begin{aligned}\mathbf{A}^2 &= \mathbf{AA} \\ &= \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} \\ &= \begin{pmatrix}\lambda^2 & 2\lambda\\ 0 & \lambda^2\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{A}^3 &= \mathbf{A}^2\mathbf{A}\\ &= \begin{pmatrix}\lambda^2 & 2\lambda\\ 0 & \lambda^2\end{pmatrix}\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\\ &= \begin{pmatrix}\lambda^3 & 3\lambda^2\\ 0 & \lambda^3\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{A}^4 &= \mathbf{A}^3\mathbf{A}\\ &= \begin{pmatrix}\lambda^3 & 3\lambda^2\\ 0 & \lambda^3\end{pmatrix}\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\\ &= \begin{pmatrix}\lambda^4 & 4\lambda^3\\ 0 & \lambda^4\end{pmatrix}\end{aligned}$$

(b) To prove

$$\mathbf{A}^n = \begin{pmatrix}\lambda^n & n\lambda^{n-1}\\ 0 & \lambda^n\end{pmatrix}$$

first note that for $n = 2$, it is true from part(a) above. Now, assume it is true for n. Then we need to show that it is true for $n+1$. For $n+1$, we have

$$\begin{aligned}\mathbf{A}^{n+1} &= \mathbf{A}^n\mathbf{A}\\ &= \begin{pmatrix}\lambda^n & n\lambda^{n-1}\\ 0 & \lambda^n\end{pmatrix}\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\\ &= \begin{pmatrix}\lambda^{n+1} & (n+1)\lambda^n\\ 0 & \lambda^{n+1}\end{pmatrix}.\end{aligned}$$

So, by induction we are done.

(c)

$$\begin{aligned}\exp(\mathbf{A}t) &= \mathbf{I} + \sum_{n=1}^{\infty}\frac{\mathbf{A}^n t^n}{n!}\\ &= \mathbf{I} + \sum_{n=1}^{\infty}\frac{t^n}{n!}\begin{pmatrix}\lambda^n & n\lambda^{n-1}\\ 0 & \lambda^n\end{pmatrix}\\ &= \begin{pmatrix}1+\sum_{n=1}^{\infty}\frac{t^n}{n!}\lambda^n & 0+\sum_{n=1}^{\infty}\frac{t^n}{n!}n\lambda^{n-1}\\ 0 & 1+\sum_{n=1}^{\infty}\frac{t^n}{n!}\lambda^n\end{pmatrix}\\ &= \begin{pmatrix}e^{\lambda t} & t+\sum_{n=2}^{\infty}\frac{t^n}{n!}n\lambda^{n-1}\\ 0 & e^{\lambda t}\end{pmatrix}\\ &= \begin{pmatrix}e^{\lambda t} & t(1+\sum_{n=2}^{\infty}\frac{t^{n-1}}{(n-1)!}\lambda^{n-1}\\ 0 & e^{\lambda t}\end{pmatrix}\\ &= \begin{pmatrix}e^{\lambda t} & te^{\lambda t}\\ 0 & e^{\lambda t}\end{pmatrix}\end{aligned}$$

Note in the above that since the sum is to ∞, if we let

$$n^* = n - 1,$$

then

$$\sum_{n=2}^{\infty} \frac{t^{n-1}}{(n-1)!}\lambda^{n-1} = \sum_{n^*=1}^{\infty} \frac{t^{n^*}}{(n^*)!}\lambda^{n^*}.$$