Problem 1 (5 pts) Solve the following initial value problem (’ represents $\frac{d}{dt}$ throughout this exam).

$$y' + ty = 2t, \quad y(0) = 1.$$  

Ans: Using the integration factor $\mu(t)$, we have

$$y'\mu + ty\mu = (y\mu)' = y'\mu + y\mu' = 2t\mu. \quad (1)$$

So, we must have

$$\mu' = t\mu.$$ 

That is,

$$\int \frac{\mu'}{\mu}dt = \int td\mu.$$ 

From this we have $\ln |\mu(t)| = \frac{t^2}{2} + c$, so let’s take $\mu(t) = e^{t^2/2}$. Now, (1) becomes:

$$(e^{t^2/2} y)' = 2te^{t^2/2}$$

Integrating this, we get

$$e^{t^2/2} y = 2e^{t^2/2} + C.$$ 

Using the initial condition, we get $C = -1$. So, finally we have

$$y(t) = 2 - e^{-t^2/2}.$$
**Problem 2** (15 pts) Consider two functions $t$ and $t^2$.

(a) (8 pts) Show that they are linearly independent on $-1 < t < 1$.

Ans: Consider the Wronskian of $t$ and $t^2$ over $(-1, 1)$.

$$W(t, t^2) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2.$$ 

Thus $W(t, t^2) \neq 0$ on $(-1, 1)$ except $t = 0$. So, these are linearly independent on $(-1, 1)$.

(b) (7 pts) Can they be a fundamental set of solutions of some differential equations $y'' + p(t)y' + q(t)y = 0$ on $-1 < t < 1$? Answer yes or no and state your reasoning clearly. [Hint: Check the Wronskian in this interval.]

Ans: They cannot be the solution of this differential equation. The reason is the following. Let $y_1, y_2$ be a fundamental set of solutions of the differential equation of the form $y'' + p(t)y' + q(t)y = 0$, where $p, q \in C(-1, 1)$. Then, from Abel’s theorem,

$$W(y_1, y_2) \neq 0, \forall t \in (-1, 1) \iff y_1, y_2 \text{ are the fundamental set of solutions.}$$

However, from (a), $W(t, t^2) = 0$ at $t = 0$. Therefore, $t$ and $t^2$ cannot be the fundamental set of solutions.
Problem 3 (15 pts) Find the general solution to the following nonhomogeneous differential equation by the method of variation of parameters.

\[ y'' + y' - 6y = t. \]

Ans: Let us first solve the homogeneous equation: \( y'' + y - 6y = 0 \). The characteristic equation is \( r^2 + r - 6 = (r - 2)(r + 3) = 0 \). Hence we have \( y_1 = e^{2t} \), and \( y_2 = e^{-3t} \). Now, consider the nonhomogeneous equation, and let the particular solution be of the following form: \( Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \). Then, \( Y' = u_1'y_1 + u_1y'_1 + u_2'y_2 + u_2y'_2 \). We choose \( u_1 \), \( u_2 \) so that \( u_1'y_1 + u_2'y_2 = 0 \). Thus, \( Y' = u_1'y_1 + u_2'y_2 \), and \( Y'' = u_1'y_1'' + u_1y'_1'' + u_2'y_2'' + u_2y'_2'' \). Inserting these to the nonhomogeneous equation, we get:

\[
Y'' + Y' - 6Y = u_1'y_1 + u_1y''_1 + u_2'y_2 + u_2y''_2 + u_1y'_1 + u_2y'_2 + 6(u_1y_1 + u_2y_2)
= u_1'y_1 + u_2'y_2 + u_1(y_1'' + y_1' - 6y_1) + u_2(y_2'' + y_2' - 6y_2)
= u_1'y_1 + u_2'y_2
= 2e^{2t}u_1' - 3e^{-3t}u_2'
= t \quad \text{the nonhomogeneous part.}
\]

Therefore, \( u_1 \) and \( u_2 \) must satisfy the following 1st order differential equations:

\[
e^{2t}u_1' + e^{-3t}u_2' = 0,
2e^{2t}u_1' - 3e^{-3t}u_2' = t,
\]

Representing these in a matrix-vector notation, we have:

\[
\begin{pmatrix}
e^{2t} & e^{-3t} \\
e^{2t} & -3e^{-3t}
\end{pmatrix}
\begin{pmatrix}
u_1' \\
u_2'
\end{pmatrix}
= \begin{pmatrix}
0 \\
t
\end{pmatrix}.
\]

That is:

\[
\begin{pmatrix}
u_1' \\
u_2'
\end{pmatrix}
= \begin{pmatrix}
e^{2t} & e^{-3t} \\
e^{2t} & -3e^{-3t}
\end{pmatrix}^{-1}
\begin{pmatrix}
0 \\
t
\end{pmatrix}
= \frac{1}{5}
\begin{pmatrix}
-3e^{-3t} & -e^{-3t} \\
-2e^{2t} & e^{2t}
\end{pmatrix}
\begin{pmatrix}
0 \\
t
\end{pmatrix}
= \frac{1}{5}
\begin{pmatrix}
t e^{-3t} \\
-2te^{2t}
\end{pmatrix}.
\]

Therefore, by integration by parts, we get:

\[
u_1(t) = -\frac{1}{20}(2t + 1)e^{-2t}, \quad u_2(t) = -\frac{1}{45}(3t - 1)e^{3t}.
\]

Hence,

\[
Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = -\frac{1}{20}(2t + 1) - \frac{1}{45}(3t - 1) = -\frac{1}{36}(6t + 1).
\]

Finally the general solution is:

\[
y(t) = c_1y_1(t) + c_2y_2(t) + Y(t) = c_1e^{2t} + c_2e^{-3t} - \frac{1}{36}(6t + 1).
\]
Problem 4 (15 pts) Solve the following initial value problem using the Laplace transform.

\[ y'' - 3y' + 2y = e^{3t}, \quad y(0) = y'(0) = \frac{1}{2}. \]

Ans: Let \( Y(s) = \mathcal{L}[y](s) \). Taking the Laplace transform of both sides yields:

\[ s^2 Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) = \frac{1}{s - 3}. \]

Putting the initial conditions, we have:

\[ (s^2 - 3s + 2)Y(s) = \frac{1}{s - 3} + \frac{s}{2} - 1. \]

Since \( s^2 - 3s + 2 = (s - 1)(s - 2) \), we have

\[ Y(s) = \frac{1}{(s - 1)(s - 2)(s - 3)} + \frac{s}{2(s - 1)(s - 2)} - \frac{1}{(s - 1)(s - 2)}. \]

Let’s represent the right-hand side by the partial fractions:

\[ \frac{A}{s - 1} + \frac{B}{s - 2} + \frac{C}{s - 3}. \]

Multiplying both sides by \((s - 1)(s - 2)(s - 3)\) and rearranging terms, we have

\[ (A + B + C)s^2 - (5A + 4B + 3C)s + 6A + 3B + 2C = \frac{1}{2}s^2 - \frac{5}{2}s + 4. \]

Comparing the coefficients of both sides, we have

\[ A = 1, \quad B = -1, \quad C = \frac{1}{2}. \]

In other words, we have:

\[ Y(s) = \frac{1}{s - 1} - \frac{1}{s - 2} + \frac{1}{2(s - 3)}. \]

Taking the inverse Laplace transform simply yields:

\[ y(t) = e^t - e^{2t} + \frac{1}{2}e^{3t}. \]
Problem 5 (15 pts) Solve the following system of differential equations.

\[
\begin{pmatrix}
x'_1 \\
x'_2
\end{pmatrix} = \begin{pmatrix}
-1 & 3 \\
2 & -2
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}.
\]

Ans: The characteristic polynomial of this matrix is:

\[
\det(rI - A) = \begin{vmatrix}
r + 1 & -3 \\
-2 & r + 2
\end{vmatrix} = (r + 1)(r + 2) - 6 = r^2 + 3r - 4 = (r - 1)(r + 4) = 0.
\]

Thus, \( r = 1, -4 \). Now let’s compute the corresponding eigenvectors.

For \( r = 1 \), we need to compute \( \xi^{(1)} \) by

\[
\begin{pmatrix}
2 & -3 \\
-2 & 3
\end{pmatrix} \begin{pmatrix}
\xi_{1}^{(1)} \\
\xi_{2}^{(1)}
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

From this, we have \( 2\xi_{1}^{(1)} - 3\xi_{2}^{(1)} = 0 \). Taking \( \xi_{1}^{(1)} = 3 \), we get \( \xi^{(1)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \).

Similarly for \( r = -4 \), we have:

\[
\begin{pmatrix}
-3 & -3 \\
-2 & -2
\end{pmatrix} \begin{pmatrix}
\xi_{1}^{(2)} \\
\xi_{2}^{(2)}
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

From this, we have \( \xi_{1}^{(2)} + \xi_{2}^{(2)} = 0 \). Taking \( \xi_{1}^{(2)} = 1 \), we get \( \xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \).

Therefore, we have the following general solution:

\[
x(t) = c_1 \xi^{(1)} e^t + c_2 \xi^{(2)} e^{-4t} = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t} = \begin{pmatrix} 3c_1 e^t + c_2 e^{-4t} \\ 2c_1 e^t - c_2 e^{-4t} \end{pmatrix}.
\]
Problem 6 (30 pts) Consider a system of differential equations:

\[ \mathbf{x}' = A \mathbf{x}, \quad A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \]

where \( \lambda \) is a real number and \( \lambda \neq 0 \).

(a) (10 pts) Use an inductive argument to show that

\[ A^k = \begin{pmatrix} \lambda^k & k \lambda^{k-1} & \frac{k(k-1)}{2} \lambda^{k-2} \\ 0 & \lambda^k & k \lambda^{k-1} \\ 0 & 0 & \lambda^k \end{pmatrix}. \]

[Hint: Induction means that you need to do the following: 1) Prove the statement is true for \( k = 0 \) and \( k = 1 \); 2) Assume the statement is true for \( k \) and prove the statement holds for \( k + 1 \).]

Ans: Let’s proceed the induction. For \( k = 0 \), we have

\[ A^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I. \]

So, this is fine.

For \( k = 1 \), we have

\[ A^1 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = A. \]

So, this is fine too.

Now, suppose this relationship is correct for \( k \). Then, consider the case \( k + 1 \).

\[ A^{k+1} = A A^k = \begin{pmatrix} \lambda^k & k \lambda^{k-1} & \frac{k(k-1)}{2} \lambda^{k-2} \\ 0 & \lambda^k & k \lambda^{k-1} \\ 0 & 0 & \lambda^k \end{pmatrix}. \]

Therefore, this formula is correct for all \( k \in \mathbb{N} \).
(b) (8 pts) Compute $e^{tA}$.

[Hint:
\[
\sum_{k=0}^{\infty} \frac{t^k}{k!} k\lambda^{k-1} = t \sum_{k=1}^{\infty} \frac{(\lambda t)^{k-1}}{(k-1)!},
\]
and
\[
\sum_{k=0}^{\infty} \frac{t^k}{k!} k(k-1)\lambda^{k-2} = \frac{t^2}{2} \sum_{k=2}^{\infty} \frac{(\lambda t)^{k-2}}{(k-2)!}.
\]
]

Ans: Recall the definition of a matrix exponential:

\[
e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}.
\]

Using (a) and the hints, we have

\[
\begin{align*}
\sum_{k=0}^{\infty} t^k &= \left( \begin{array}{ccc}
\lambda^k & k\lambda^{k-1} & \frac{k(k-1)}{2}\lambda^{k-2} \\
0 & \lambda^k & k\lambda^{k-1} \\
0 & 0 & \lambda^k
\end{array} \right) \\
&= \left( \begin{array}{ccc}
\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} & \sum_{k=1}^{\infty} \frac{t(\lambda t)^{k-1}}{(k-1)!} & \sum_{k=2}^{\infty} \frac{t^2 (\lambda t)^{k-2}}{2(k-2)!} \\
0 & \sum_{k=0}^{\infty} \frac{t(\lambda t)^k}{k!} & \sum_{k=1}^{\infty} \frac{t(\lambda t)^{k-1}}{(k-1)!} \\
0 & 0 & \sum_{k=0}^{\infty} \frac{t(\lambda t)^k}{k!}
\end{array} \right) \\
&= \left( \begin{array}{ccc}
e^\lambda t & t e^\lambda t & \frac{t^2}{2} e^\lambda t \\
0 & e^\lambda t & t e^\lambda t \\
0 & 0 & e^\lambda t
\end{array} \right)
\end{align*}
\]
(c) (6 pts) Find the fundamental matrix $\Phi(t)$ satisfying $\Phi(1) = I$. Note that $\Phi(1) = I$.
This is different from $\Phi(0) = I$.
Ans: For this initial condition, it is clear that
$$\Phi(t) = e^{(t-1)A}$$
Using (b), we clearly have:
$$\Phi(t) = e^{(t-1)A} = \begin{pmatrix}
e^{\lambda (t-1)} & (t-1)e^{\lambda (t-1)} & \frac{(t-1)^2}{2} e^{\lambda (t-1)} \\
0 & e^{\lambda (t-1)} & (t-1)e^{\lambda (t-1)} \\
0 & 0 & e^{\lambda (t-1)}
\end{pmatrix}.$$ 
It is easy to verify that $\Phi(1) = I$. 
(d) (6 pts) Suppose that the initial condition:

\[ \mathbf{x}(1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \]

is given. Then, solve this IVP.

**Ans:** Using the fundamental matrix \( \Phi(t) \), we have the following solution: \( \mathbf{x}(t) = \Phi(t) \mathbf{x}(1) \). Therefore,

\[
\mathbf{x}(t) = \begin{pmatrix}
    e^{\lambda(t-1)} & (t - 1)e^{\lambda(t-1)} & \frac{(t-1)^2}{2}e^{\lambda(t-1)} \\
    0 & e^{\lambda(t-1)} & (t - 1)e^{\lambda(t-1)} \\
    0 & 0 & e^{\lambda(t-1)}
\end{pmatrix}
\begin{pmatrix}
    1 \\
    1 \\
    1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    (1 + (t - 1) + \frac{(t-1)^2}{2})e^{\lambda(t-1)} \\
    (1 + (t - 1))e^{\lambda(t-1)} \\
    e^{\lambda(t-1)}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    \frac{t^2+1}{2}e^{\lambda(t-1)} \\
    te^{\lambda(t-1)} \\
    e^{\lambda(t-1)}
\end{pmatrix}.
\]
Problem 7  (20 pts) Solve the following system of differential equations.

\[
\begin{pmatrix}
  x'_1 \\
  x'_2
\end{pmatrix} = \begin{pmatrix}
  -8 & 4 \\
  -9 & 4
\end{pmatrix} \begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}.
\]

Ans: The characteristic polynomial of this matrix is:

\[
\begin{vmatrix}
  r + 8 & -4 \\
  9 & r - 4
\end{vmatrix} = r^2 + 4r + 4 = (r + 2)^2 = 0.
\]

So, we have repeated eigenvalues \( r = -2 \). Let’s compute the eigenvector \( \xi^{(1)} \).

\[
\begin{pmatrix}
  6 & 4 \\
  9 & -6
\end{pmatrix} \begin{pmatrix}
  \xi_{1}^{(1)} \\
  \xi_{2}^{(1)}
\end{pmatrix} = \begin{pmatrix}
  0 \\
  0
\end{pmatrix}.
\]

From this, we have \( 6\xi_{1}^{(1)} - 4\xi_{2}^{(1)} = 0 \). Taking \( \xi_{1}^{(1)} = 2 \), we get \( \xi^{(1)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \). Thus,

\[ x^{(1)} = \xi^{(1)} e^{-2t} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}. \]

To compute \( x^{(2)} \), let us assume that \( x^{(2)} = t\xi^{(1)} e^{-2t} + \xi^{(2)} e^{-2t} \). The vector \( \xi^{(2)} \) must satisfy the following equation: \( (A - rI)\xi^{(2)} = \xi^{(1)} \), i.e.,

\[
\begin{pmatrix}
  -6 & 4 \\
  -9 & 6
\end{pmatrix} \begin{pmatrix}
  \xi_{1}^{(2)} \\
  \xi_{2}^{(2)}
\end{pmatrix} = \begin{pmatrix}
  2 \\
  3
\end{pmatrix}.
\]

From this, we have \( -6\xi_{1}^{(2)} + 4\xi_{2}^{(2)} = 2 \). Taking \( \xi_{1}^{(2)} = -1 \), we get \( \xi^{(2)} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \). Here, \( \xi_{1}^{(2)} \) can take other values. For example, the other choices lead to \( \xi^{(2)} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \) or \( \xi^{(2)} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \).

Thus, the general solution is:

\[
y(t) = c_1 x^{(1)}(t) + c_2 x^{(2)} = c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} t + \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}
\]

\[
= \begin{pmatrix} 2c_1 + (2c_2t - c_2) \\ 3c_1 + (3c_2t - c_2) \end{pmatrix} e^{-2t}
\]

\[
= \begin{pmatrix} 2c_1 - c_2 + 2c_2t \\ 3c_1 - c_2 + 3c_2t \end{pmatrix} e^{-2t}.
\]
**Bonus Problem** (20 pts) Show that all solutions of \( ay'' + by' + cy = 0 \) approach zero as \( t \to \infty \) if \( a, b, c \) are all positive real constants.

[Hint: Think the graph of \( ar^2 + br + c \) (a parabola) when \( a, b, c \) are positive. There are three cases you need to check.]

Ans: The characteristic equation is clearly \( ar^2 + br + c = 0 \). Let \( \lambda \) and consider the roots of \( f(\lambda) = 0 \).

**Case I:** \( D = b^2 - 4ac > 0 \), i.e., two distinct real roots. In this case, because \( c > 0 \), \( b > 0 \), \( a > 0 \), both roots are negative. (Consider the graph of \( f(\lambda) \) which is a convex parabola.)

Let these two roots be \( r = \alpha \) and \( r = \beta \) with \( \alpha < 0 \) and \( \beta < 0 \). Then, the general solution is:

\[
y(t) = c_1e^{\alpha t} + c_2e^{\beta t}.
\]

No matter what values \( c_1 \) and \( c_2 \) take, both \( e^{\alpha t} \) and \( e^{\beta t} \) tend to 0 as \( t \to \infty \) because \( \alpha < 0 \) and \( \beta < 0 \). Thus, \( y(t) \to 0 \) as \( t \to \infty \).

**Case II:** \( D = b^2 - 4ac = 0 \). In this case, we have repeated roots \( r = -b/2a \), which is negative since \( a > 0 \) and \( b > 0 \). The graph of \( f(\lambda) \) in this case is a parabola whose apex is touching on \( r \)-axis at \( r = -b/2a \). The general solution in this case is:

\[
y(t) = c_1e^{-bt/2a} + c_2te^{-bt/2a}.
\]

But both \( e^{-bt/2a} \) and \( te^{-bt/2a} \) tend to 0 as \( t \to \infty \) since \( -b/2a < 0 \) and \( e^{bt/2a} \) grows much faster than \( t \). Thus, \( y(t) \to 0 \) as \( t \to \infty \).

**Case III:** \( D = b^2 - 4ac < 0 \). In this case, we have two complex roots,

\[
r = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}
\]

So, we have the general solution:

\[
y(t) = c_1e^{-bt/2a} \cos \left( \frac{\sqrt{4ac - b^2}}{2a} t \right) + c_2e^{-bt/2a} \sin \left( \frac{\sqrt{4ac - b^2}}{2a} t \right).
\]

Again, because of the factor \( e^{-bt/2a} \), \( y(t) \to 0 \) as \( t \to \infty \). (The solution oscillates but dies out eventually.)

Therefore, in all cases, we established \( y(t) \to 0 \) as \( t \to \infty \).
### Table of Elementary Laplace Transforms

<table>
<thead>
<tr>
<th>$f(t) = \mathcal{L}^{-1}(F(s))$</th>
<th>$F(s) = \mathcal{L}(f(t))$</th>
</tr>
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<tr>
<td>1</td>
<td>$\frac{1}{s}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$t^n$, $n \in \mathbb{N}$</td>
<td>$\frac{n!}{s^{n+1}}$, $s &gt; 0$</td>
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<td>$e^{at}$</td>
<td>$\frac{1}{s-a}$, $s &gt; a$</td>
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<td>$\cos at$</td>
<td>$\frac{s}{s^2 + a^2}$, $s &gt; 0$</td>
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<tr>
<td>$\sin at$</td>
<td>$\frac{a}{s^2 + a^2}$, $s &gt; 0$</td>
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