MAT 22B-001: Differential Equations
Final Exam Solutions

Note: There is a table of the Laplace transform in the last page.

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Problem 1 (5 pts) Solve the following initial value problem ($'$ represents $\frac{\mathrm{d}}{\mathrm{d}t}$ throughout this exam).

$$y' + ty = 2t, \quad y(0) = 1.$$

Ans: Using the integration factor $\mu(t)$, we have

$$y'\mu + ty\mu = (y\mu)' = y'\mu + y\mu' = 2t\mu. \tag{1}$$

So, we must have

$$\mu' = t\mu.$$

That is,

$$\int \frac{\mu'}{\mu}\mathrm{d}t = \int t\mathrm{d}t.$$

From this we have $\ln|\mu(t)| = t^2/2 + c$, so let's take $\mu(t) = \mathrm{e}^{t^2/2}$. Now, (1) becomes:

$$(\mathrm{e}^{t^2/2}y)' = 2t\mathrm{e}^{t^2/2}$$

Integrating this, we get

$$\mathrm{e}^{t^2/2}y = 2\mathrm{e}^{t^2/2} + C.$$

Using the initial condition, we get $C = -1$. So, finally we have

$$y(t) = 2 - \mathrm{e}^{-t^2/2}.$$

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Problem 2 (15 pts) Consider two functions t and t^2.

(a) (8 pts) Show that they are linearly independent on $-1 < t < 1$.
Ans: Consider the Wronskian of t and t^2 over $(-1, 1)$.

$$W(t, t^2) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2.$$

Thus $W(t, t^2) \neq 0$ on $(-1, 1)$ except $t = 0$. So, these are linearly independent on $(-1, 1)$.

(b) (7 pts) Can they be a fundamental set of solutions of some differential equations $y'' + p(t)y' + q(t)y = 0$ on $-1 < t < 1$? Answer yes or no and state your reasoning clearly. [Hint: Check the Wronskian in this interval.]

Ans: They cannot be the solution of this differential equation. The reason is the following. Let y_1, y_2 be a fundamental set of solutions of the differential equation of the form $y'' + p(t)y' + q(t)y = 0$, where $p, q \in C(-1, 1)$. Then, from Abel's theorem,

$$W(y_1, y_2) \neq 0,\ \forall t \in (-1, 1) \iff y_1, y_2 \text{ are the fundamental set of solutions.}$$

However, from (a), $W(t, t^2) = 0$ at $t = 0$. Therefore, t and t^2 cannot be the fundamental set of solutions.

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Problem 3 (15 pts) Find the general solution to the following nonhomogeneous differential equation by the method of *variation of parameters*.

$$y'' + y' - 6y = t.$$

Ans: Let us first solve the homogeneous equation: $y'' + y - 6y = 0$. The characteristic equation is $r^2 + r - 6 = (r-2)(r+3) = 0$. Hence we have $y_1 = \mathrm{e}^{2t}$, and $y_2 = \mathrm{e}^{-3t}$. Now, consider the nonhomogeneous equation, and let the particular solution be of the following form: $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$. Then, $Y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$. We choose u_1, u_2 so that $u_1'y_1 + u_2'y_2 = 0$. Thus, $Y' = u_1y_1' + u_2y_2'$, and $Y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$. Inserting these to the nonhomogeneous equation, we get:

$$\begin{aligned}
Y'' + Y' - 6Y &= u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' + u_1y_1' + u_2y_2' + 6(u_1y_1 + u_2y_2) \\
&= u_1'y_1' + u_2'y_2' + u_1(y_1'' + y_1' - 6y_1) + u_2(y_2'' + y_2' - 6y_2) \\
&= u_1'y_1' + u_2'y_2' \\
&= 2\mathrm{e}^{2t}u_1' - 3\mathrm{e}^{-3t}u_2' \\
&= t \quad \text{the nonhomogeneous part.}
\end{aligned}$$

Therefore, u_1 and u_2 must satisfy the following 1st order differential equations:

$$\mathrm{e}^{2t}u_1' + \mathrm{e}^{-3t}u_2' = 0,$$

$$2\mathrm{e}^{2t}u_1' - 3\mathrm{e}^{-3t}u_2' = t,$$

Representing these in a matrix-vector notation, we have:

$$\begin{pmatrix} \mathrm{e}^{2t} & \mathrm{e}^{-3t} \\ 2\mathrm{e}^{2t} & -3\mathrm{e}^{-3t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ t \end{pmatrix}.$$

That is:

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{2t} & \mathrm{e}^{-3t} \\ 2\mathrm{e}^{2t} & -3\mathrm{e}^{-3t} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ t \end{pmatrix} = -\frac{\mathrm{e}^{t}}{5} \begin{pmatrix} -3\mathrm{e}^{-3t} & -\mathrm{e}^{-3t} \\ -2\mathrm{e}^{2t} & \mathrm{e}^{2t} \end{pmatrix} \begin{pmatrix} 0 \\ t \end{pmatrix} = \frac{1}{5} \begin{pmatrix} t\mathrm{e}^{-2t} \\ -t\mathrm{e}^{3t} \end{pmatrix}.$$

Therefore, by integration by parts, we get:

$$u_1(t) = -\frac{1}{20}(2t+1)\mathrm{e}^{-2t}, \quad u_2(t) = -\frac{1}{45}(3t-1)\mathrm{e}^{3t}.$$

Hence,

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = -\frac{1}{20}(2t+1) - \frac{1}{45}(3t-1) = -\frac{1}{36}(6t+1).$$

Finally the general solution is:

$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t) = c_1\mathrm{e}^{2t} + c_2\mathrm{e}^{-3t} - \frac{1}{36}(6t+1).$$

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Problem 4 (15 pts) Solve the following initial value problem using the *Laplace transform*.

$$y'' - 3y' + 2y = \mathrm{e}^{3t}, \quad y(0) = y'(0) = \frac{1}{2}.$$

Ans: Let $Y(s) = \mathcal{L}[y](s)$. Taking the Laplace transform of both sides yields:

$$s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) = \frac{1}{s-3}.$$

Putting the initial conditions, we have:

$$(s^2 - 3s + 2)Y(s) = \frac{1}{s-3} + \frac{s}{2} - 1.$$

Since $(s^2 - 3s + 2) = (s-1)(s-2)$, we have

$$Y(s) = \frac{1}{(s-1)(s-2)(s-3)} + \frac{s}{2(s-1)(s-2)} - \frac{1}{(s-1)(s-2)}.$$

Let's represent the right-hand side by the partial fractions:

$$\frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}.$$

Multiplying both sides by $(s-1)(s-2)(s-3)$ and rearranging terms, we have

$$(A+B+C)s^2 - (5A+4B+3C)s + 6A + 3B + 2C = \frac{1}{2}s^2 - \frac{5}{2}s + 4.$$

Comparing the coefficients of both sides, we have

$$A = 1,\ B = -1,\ C = \frac{1}{2}.$$

In other words, we have:

$$Y(s) = \frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{2(s-3)}.$$

Taking the inverse Laplace transform simply yields:

$$y(t) = \mathrm{e}^{t} - \mathrm{e}^{2t} + \frac{1}{2}\mathrm{e}^{3t}.$$

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Problem 5 (15 pts) Solve the following system of differential equations.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Ans: The characteristic polynomial of this matrix is:

$$\det(rI - A) = \begin{vmatrix} r+1 & -3 \\ -2 & r+2 \end{vmatrix} = (r+1)(r+2) - 6 = r^2 + 3r - 4 = (r-1)(r+4) = 0.$$

Thus, $r = 1, -4$. Now let's compute the corresponding eigenvectors.

For $r = 1$, we need to compute $\boldsymbol{\xi}^{(1)}$ by

$$\begin{pmatrix} 2 & -3 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From this, we have $2\xi_1^{(1)} - 3\xi_2^{(1)} = 0$. Taking $\xi_1^{(1)} = 3$, we get $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Similarly for $r = -4$, we have:

$$\begin{pmatrix} -3 & -3 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From this, we have $\xi_1^{(2)} + \xi_2^{(2)} = 0$. Taking $\xi_1^{(2)} = 1$, we get $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Therefore, we have the following general solution:

$$\begin{aligned} \boldsymbol{x}(t) &= c_1\boldsymbol{\xi}^{(1)}\mathrm{e}^t + c_2\boldsymbol{\xi}^{(2)}\mathrm{e}^{-4t} \\ &= c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \mathrm{e}^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mathrm{e}^{-4t} \\ &= \begin{pmatrix} 3c_1\mathrm{e}^t + c_2\mathrm{e}^{-4t} \\ 2c_1\mathrm{e}^t - c_2\mathrm{e}^{-4t} \end{pmatrix}. \end{aligned}$$

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Problem 6 (30 pts) Consider a system of differential equations:

$$\boldsymbol{x}' = A\boldsymbol{x}, \quad A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

where λ is a real number and $\lambda \neq 0$.

(a) (10 pts) Use an inductive argument to show that

$$A^k = \begin{pmatrix} \lambda^k & k\lambda^{k-1} & \frac{k(k-1)}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{pmatrix}.$$

[Hint: Induction means that you need to do the following: 1) Prove the statement is true for $k = 0$ and $k = 1$; 2) Assume the statement is true for k and prove the statement holds for $k + 1$.]

Ans: Let's proceed the induction. For $k = 0$, we have

$$A^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

So, this is fine.

For $k = 1$, we have

$$A^1 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = A.$$

So, this is fine too.

Now, suppose this relationship is correct for k. Then, consider the case $k + 1$.

$$\begin{aligned} A^{k+1} &= A\,A^k \\ &= \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} \lambda^k & k\lambda^{k-1} & \frac{k(k-1)}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{pmatrix} \\ &= \begin{pmatrix} \lambda^{k+1} & (k+1)\lambda^k & \frac{k(k-1)}{2}\lambda^{k-1} + k\lambda^{k-1} \\ 0 & \lambda^{k+1} & (k+1)\lambda^k \\ 0 & 0 & \lambda^{k+1} \end{pmatrix} \\ &= \begin{pmatrix} \lambda^{k+1} & (k+1)\lambda^k & \frac{k(k+1)}{2}\lambda^{k-1} \\ 0 & \lambda^{k+1} & (k+1)\lambda^k \\ 0 & 0 & \lambda^{k+1} \end{pmatrix} \end{aligned}$$

Therefore, this formula is correct for all $k \in \mathbb{N}$.

(b) (8 pts) Compute e^{tA}.

[Hint:

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} k\lambda^{k-1} = t \sum_{k=1}^{\infty} \frac{(\lambda t)^{k-1}}{(k-1)!},$$

and

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{k(k-1)}{2} \lambda^{k-2} = \frac{t^2}{2} \sum_{k=2}^{\infty} \frac{(\lambda t)^{k-2}}{(k-2)!}.$$

]

Ans: Recall the definition of a matrix exponential:

$$\mathrm{e}^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}.$$

Using (a) and the hints, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{t^k}{k!} &= \begin{pmatrix} \lambda^k & k\lambda^{k-1} & \frac{k(k-1)}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{pmatrix} \\
&= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} & \sum_{k=1}^{\infty} t\frac{(\lambda t)^{k-1}}{(k-1)!} & \sum_{k=2}^{\infty} \frac{t^2}{2} \frac{(\lambda t)^{k-2}}{(k-2)!} \\ 0 & \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} & \sum_{k=1}^{\infty} t\frac{(\lambda t)^{k-1}}{(k-1)!} \\ 0 & 0 & \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{e}^{\lambda t} & t\mathrm{e}^{\lambda t} & \frac{t^2}{2}\mathrm{e}^{\lambda t} \\ 0 & \mathrm{e}^{\lambda t} & t\mathrm{e}^{\lambda t} \\ 0 & 0 & \mathrm{e}^{\lambda t} \end{pmatrix}
\end{aligned}$$

(c) (6 pts) Find the fundamental matrix $\Phi(t)$ satisfying $\Phi(1) = I$. Note that $\Phi(1) = I$; This is different from $\Phi(0) = I$.

Ans: For this initial condition, it is clear that

$$\Phi(t) = \mathrm{e}^{(t-1)A}$$

Using (b), we clearly have:

$$\Phi(t) = \mathrm{e}^{(t-1)A} = \begin{pmatrix} \mathrm{e}^{\lambda(t-1)} & (t-1)\mathrm{e}^{\lambda(t-1)} & \frac{(t-1)^2}{2}\mathrm{e}^{\lambda(t-1)} \\ 0 & \mathrm{e}^{\lambda(t-1)} & (t-1)\mathrm{e}^{\lambda(t-1)} \\ 0 & 0 & \mathrm{e}^{\lambda(t-1)} \end{pmatrix}.$$

It is easy to verify that $\Phi(1) = I$.

(d) (6 pts) Suppose that the initial condition:

$$\boldsymbol{x}(1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

is given. Then, solve this IVP.

Ans: Using the fundamental matrix $\Phi(t)$, we have the following solution: $\boldsymbol{x}(t) = \Phi(t)\boldsymbol{x}(1)$. Therefore,

$$\begin{aligned} \boldsymbol{x}(t) &= \begin{pmatrix} \mathrm{e}^{\lambda(t-1)} & (t-1)\mathrm{e}^{\lambda(t-1)} & \frac{(t-1)^2}{2}\mathrm{e}^{\lambda(t-1)} \\ 0 & \mathrm{e}^{\lambda(t-1)} & (t-1)\mathrm{e}^{\lambda(t-1)} \\ 0 & 0 & \mathrm{e}^{\lambda(t-1)} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \left(1 + (t-1) + \frac{(t-1)^2}{2}\right) \mathrm{e}^{\lambda(t-1)} \\ (1 + (t-1))\,\mathrm{e}^{\lambda(t-1)} \\ \mathrm{e}^{\lambda(t-1)} \end{pmatrix} \\ &= \begin{pmatrix} \frac{t^2+1}{2}\mathrm{e}^{\lambda(t-1)} \\ t\mathrm{e}^{\lambda(t-1)} \\ \mathrm{e}^{\lambda(t-1)} \end{pmatrix}. \end{aligned}$$

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Problem 7 (20 pts) Solve the following system of differential equations.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -8 & 4 \\ -9 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Ans: The characteristic polynomial of this matrix is:

$$\begin{vmatrix} r+8 & -4 \\ 9 & r-4 \end{vmatrix} = r^2 + 4r + 4 = (r+2)^2 = 0.$$

So, we have repeated eigenvalues $r = -2$. Let's compute the eigenvector $\boldsymbol{\xi}^{(1)}$.

$$\begin{pmatrix} 6 & 4 \\ 9 & -6 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From this, we have $6\xi_1^{(1)} - 4\xi_2^{(1)} = 0$. Taking $\xi_1^{(1)} = 2$, we get $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Thus,

$$\boldsymbol{x}^{(1)} = \boldsymbol{\xi}^{(1)} \mathrm{e}^{-2t} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \mathrm{e}^{-2t}.$$

To compute $\boldsymbol{x}^{(2)}$, let us assume that $\boldsymbol{x}^{(2)} = t\boldsymbol{\xi}^{(1)}\mathrm{e}^{-2t} + \boldsymbol{\xi}^{(2)}\mathrm{e}^{-2t}$. The vector $\boldsymbol{\xi}^{(2)}$ must satisfy the following equation: $(A - rI)\boldsymbol{\xi}^{(2)} = \boldsymbol{\xi}^{(1)}$, i.e.,

$$\begin{pmatrix} -6 & 4 \\ -9 & 6 \end{pmatrix} \begin{pmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{pmatrix} = \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

From this, we have $-6\xi_1^{(2)} + 4\xi_2^{(2)} = 2$. Taking $\xi_1^{(2)} = -1$, we get $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. Here, $\xi_1^{(2)}$ can take other values. For example, the other choices lead to $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$ or $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$. Thus, the general solution is:

$$\begin{aligned} y(t) &= c_1 \boldsymbol{x}^{(1)}(t) + c_2 \boldsymbol{x}^{(2)} \\ &= c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} \mathrm{e}^{-2t} + c_2 \left[\begin{pmatrix} 2 \\ 3 \end{pmatrix} t + \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right] \mathrm{e}^{-2t} \\ &= \begin{pmatrix} 2c_1 + (2c_2 t - c_2) \\ 3c_1 + (3c_2 t - c_2) \end{pmatrix} \mathrm{e}^{-2t} \\ &= \begin{pmatrix} 2c_1 - c_2 + 2c_2 t \\ 3c_1 - c_2 + 3c_2 t \end{pmatrix} \mathrm{e}^{-2t}. \end{aligned}$$

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Bonus Problem (20 pts) Show that all solutions of $ay'' + by' + cy = 0$ approach zero as $t \to \infty$ if a, b, c are all positive real constants.

[Hint: Think the graph of $ar^2 + br + c$ (a parabola) when a, b, c are positive. There are three cases you need to check.]

Ans: The characteristic equation is clearly $ar^2 + br + c = 0$. Let $f(r) = ar^2 + br + c$ and consider the roots of $f(r) = 0$.

Case I: $D = b^2 - 4ac > 0$, i.e., two distinct real roots. In this case, because $c > 0$, $b > 0$, $a > 0$, both roots are negative. (Consider the graph of $f(r)$ which is a convex parabola.) Let these two roots be $r = \alpha$ and $r = \beta$ with $\alpha < 0$ and $\beta < 0$. Then, the general solution is:

$$y(t) = c_1 \mathrm{e}^{\alpha t} + c_2 \mathrm{e}^{\beta t}.$$

No matter what values c_1 and c_2 take, both $\mathrm{e}^{\alpha t}$ and $\mathrm{e}^{\beta t}$ tend to 0 as $t \to \infty$ because $\alpha < 0$ and $\beta < 0$. Thus, $y(t) \to 0$ as $t \to \infty$.

Case II: $D = b^2 - 4ac = 0$. In this case, we have repeated roots $r = -b/2a$, which is negative since $a > 0$ and $b > 0$. The graph of $f(r)$ in this case is a parabola whose apex is touching on r-axis at $r = -b/2a$. The general solution in this case is:

$$y(t) = c_1 \mathrm{e}^{-bt/2a} + c_2 t \mathrm{e}^{-bt/2a}.$$

But both $\mathrm{e}^{-bt/2a}$ and $t\mathrm{e}^{-bt/2a}$ tend to 0 as $t \to \infty$ since $-b/2a < 0$ and $\mathrm{e}^{bt/2a}$ grows much faster than t. Thus, $y(t) \to 0$ as $t \to \infty$.

Case III: $D = b^2 - 4ac < 0$. In this case, we have two complex roots,

$$r = \frac{-b \pm \mathrm{i}\sqrt{4ac - b^2}}{2a}$$

So, we have the general solution:

$$y(t) = c_1 \mathrm{e}^{-bt/2a} \cos\left(\frac{\sqrt{4ac - b^2}}{2a} t\right) + c_2 \mathrm{e}^{-bt/2a} \sin\left(\frac{\sqrt{4ac - b^2}}{2a} t\right).$$

Again, because of the factor $\mathrm{e}^{-bt/2a}$, $y(t) \to 0$ as $t \to \infty$. (The solution oscillates but dies out eventually.)

Therefore, in all cases, we established $y(t) \to 0$ as $t \to \infty$.

Table of Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}(F(s))$	$F(s) = \mathcal{L}(f(t))$
1	$\frac{1}{s}, \quad s > 0$
$t^n, n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$