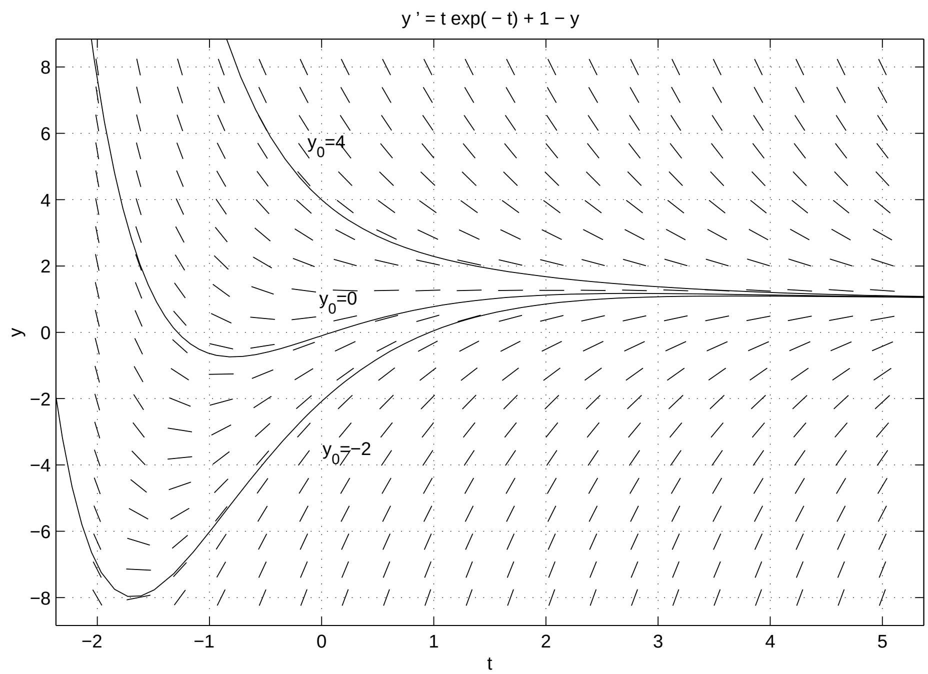
§2.1

#3Draw a direction field for the equation $y' + y = e^{-t} + 1, \quad y(0) = y_0$



We can solve the ODE using an integrating factor. If we set $\mu(t) = e^t$ and multiply both sides of the equation by μ we get:

$$y'e^t + ye^t = (te^{-t} + 1)e^{-t} = t + e^t$$

thus the new equation is:

$$y'e^t + ye^t = (ye^t)' = t + e^t$$

So, after integrating we get:

$$ye^t = \frac{t^2}{2} + e^t + C$$

Then we solve for $y(t)$ to get

$$y(t) = [\frac{t^2}{2} + (1 - y_0)]e^{-t} + 1$$

since $C = 1 - y_0$.

#13:Solve the IVP $y' - y = 2te^{2t}, \quad y(0) = 1$ using an integrating factor. Again let $\mu(t)$ be the integrating factor and choose $\mu(t) = e^{-t}$. Multiplying the equation be μ then integrating yields:

$$ye^{-t} = \int (2te^{2t})e^{-t}dt = 2\int te^t dt$$

but

$$\int te^t dt = te^t - e^t + C$$

so we have

$$ye^{-t} = 2te^t - 2e^t + K$$

where $K = 2C$. Solving for $y(t)$ gives:

$$y(t) = 2te^{2t} - 2e^{2t} + Ke^t$$

Then, using the initial condition to solve for K ($K = 0$) we get:

$$y(t) = 2te^{2t} - 2e^{2t}$$

#29:Consider the IVP $y' - \frac{3}{2}y = 3t + 2e^t, \quad y(0) = y_0$. Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. Again we use an integrating factor to solve the IVP in terms of y_0 then take limits. The integrating factor is $\mu(t) = e^{-\frac{3}{2}t}$. Then if we let $g(t) = 3t + 2e^t$ and using the formula:

$$y(t) = \frac{1}{\mu(t)} \int g(t)\mu(t)dt$$

we get:

$$y(t) = Ke^{\frac{3}{2}t} - 4e^t - 2t - \frac{4}{3}$$

where $K = y_0 + \frac{16}{3}$. Now if we let $t \to \infty$ we get:

$$lim_{t\to\infty} y(t) = \begin{cases} +\infty & K > 0 \\ -\infty & K \leq 0 \end{cases}$$

Thus we have:

$$lim_{t\to\infty} y(t) = \begin{cases} +\infty & y_0 > -\frac{16}{3} \\ -\infty & y_0 \leq -\frac{16}{3} \end{cases}$$

#35:Consider the general 1^{st} order, linear, ODE: $y' + p(t)y = g(t)$. (1)
a)If $g(t) \equiv 0$ then we have $y' + p(t)y = 0$. This equation is easily solved using seperation of variables. We get:

$$y(t) = Ae^{-\int p(t)dt}$$

where A is a constant.
b)If $g(t) \neq 0$ we look for solutions of the form $y(t) = A(t)e^{-\int p(t)dt}$ where the coefficient of the exponential, $A(t)$, is no longer a constant, but a function of t. If we differentiate we get:

$$y'(t) = A'(t)e^{-\int p(t)} - A(t)p(t)e^{-\int p(t)}$$

subsituting this form into (1) we get:

$$y' + py = (A'e^{-\int pdt} - Ape^{-\int pdt}) + pAe^{-\int pdt} = g$$

or

$$A'(t) = g(t)e^{\int p(t)dt}$$

c)Solving the above equation for $A(t)$ we get:

$$A(t) = \int g(t)e^{\int p(t)dt}dt$$

thus if we substitute this back into the assumed form of $y(t)$ we have:

$$y(t) = A(t)e^{-\int p(t)dt} = e^{-\int p(t)dt}\int g(t)e^{\int p(t)dt}dt$$

Now if we let $\mu(t) = e^{\int p(t)dt}$ then the above equation becomes:

$$y(t) = \frac{1}{\mu(t)}\int g(t)\mu(t)dt$$

which is the general solution to (1) obtained by using an integrating factor.

§2.2

#5: Solve the ODE $y' = \cos^2 x \cos^2 2y$. We use separation of variables and get:

$$\frac{dy}{\cos^2 2y} = \cos^2 x dx$$

thus integrating both sides leaves:

$$\frac{1}{2}\tan 2y = \int \cos^2 x dx = \frac{1}{2}\int (\cos 2x + 1)dx$$

(Note the use of the identity $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$. Remember it!) After integrating the left hand side we have:

$$\tan 2y = \frac{1}{2}\sin 2x + 2x + K$$

#8:Solve the ODE $\frac{dy}{dx} = \frac{x^2}{y^2+1}$. Again we use separation of variables to get:

$$(y^2 + 1)dy = x^2 dx$$

Integrating yields:

$$\frac{y^3}{3} + y - \frac{x^3}{3} = K$$

#12:Solve the IVP $\frac{dr}{d\theta} = \frac{r^2}{\theta} \quad r(1) = 2$. Separating variable leaves:

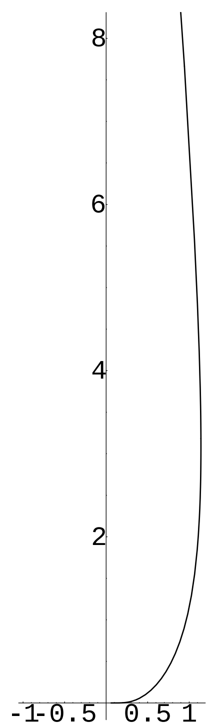
$$\frac{dr}{r^2} = \frac{d\theta}{\theta}$$

Integrating and solving for $r(\theta)$ gives:

$$r(\theta) = -\frac{1}{\ln\theta + K}$$

Solving for the initial condition gives $K = -\frac{1}{2}$, thus we have:

$$r(\theta) = -\frac{1}{\ln\theta - \frac{1}{2}}$$



The interval of definition is $0 < \theta < \sqrt{e}$

#23:Solve the IVP $y' = 2y^2 + xy^2 \quad y(0) = 1$. The problem is non-linear, so the only technique we have learned that can be applied is Separation of Variables. Thus we separate:

$$\frac{dy}{y^2} = (2 + x)dx$$

integrate:

$$-\frac{1}{y} = \frac{x^2}{2} + 2x + K$$

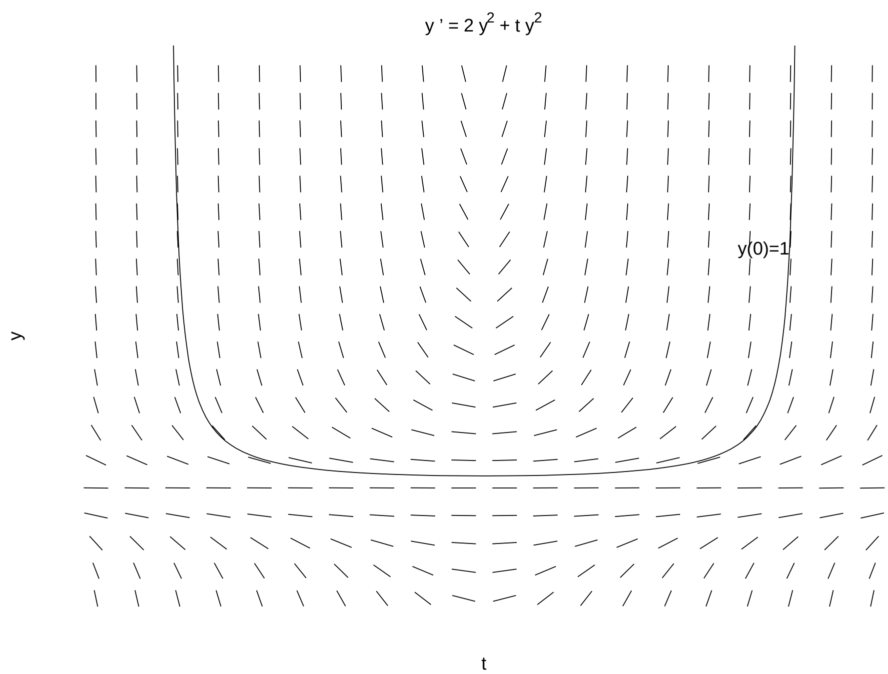
and finally solve for $y(x)$ with the given initial conditions:

$$y(x) = -\frac{2}{x^2 + 4x - 2} \qquad (-2 - \sqrt{6}) < x < (-2 + \sqrt{6})$$

Then, to find the minimum we differntiate and find the critical points to get $x = -2$. This is in the domain of our solution, but we must check that it is a local-min. But $y'(-3) < 0$ and $y'(-1) > 0$, so $x = -2$ is a local-min. Moreover, we have:

$$lim_{x \to (-2 \pm \sqrt{6})} = +\infty$$

so $x = -2$ is an absolute-min.



§2.3

#4:To set up the differential equation let $A(t) =$"The amount (in poundss) of chemical in the tank at time t". Also in this problem the volume of solution is not fixed, so let $V(t) =$"The volume (in gallons) of solution in the tank at time t".Then we have the following:

$$\frac{dA}{dt} = Rate\ of\ change\ of\ the\ amount\ of\ chemical\ in\ the\ tank.$$

$$Rate\ of\ Change = (Rate\ In) - (Rate\ Out)$$

$$Rate\ In = 1\frac{lb}{gal} \times 3\frac{gal}{min}$$

$$Rate\ Out = \frac{A(t)}{V(t)}\frac{lb}{gal} \times 2\frac{gal}{min}$$

The term $\frac{A(t)}{V(t)}$ is the concentration of the solution leaving the tank. But $V(t)$ is given as $V(t) = 200 + t$, so:

$$Rate\ Out = \frac{A(t)}{200+t}\frac{lb}{gal} \times 2\frac{gal}{min}$$

thus we have the following differential equation:

$$\frac{dA}{dt} = \left(1\frac{lb}{gal} \times 3\frac{gal}{min}\right) - \left(\frac{A(t)}{200+t}\frac{lb}{gal} \times 2\frac{gal}{min}\right)$$

$$= 3 - \frac{2A}{200+t}$$

so the IVP we must solve is:

$$\frac{dA}{dt} + \frac{2A}{200+t} = 3 \qquad A(0) = 100$$

To solve this we use and integrating factor $\mu(t) = e^{\int(\frac{2}{200+t})dt} = (200+t)^2$. Thus if we take $g(t) = 3$ the general solution of the above equation is:

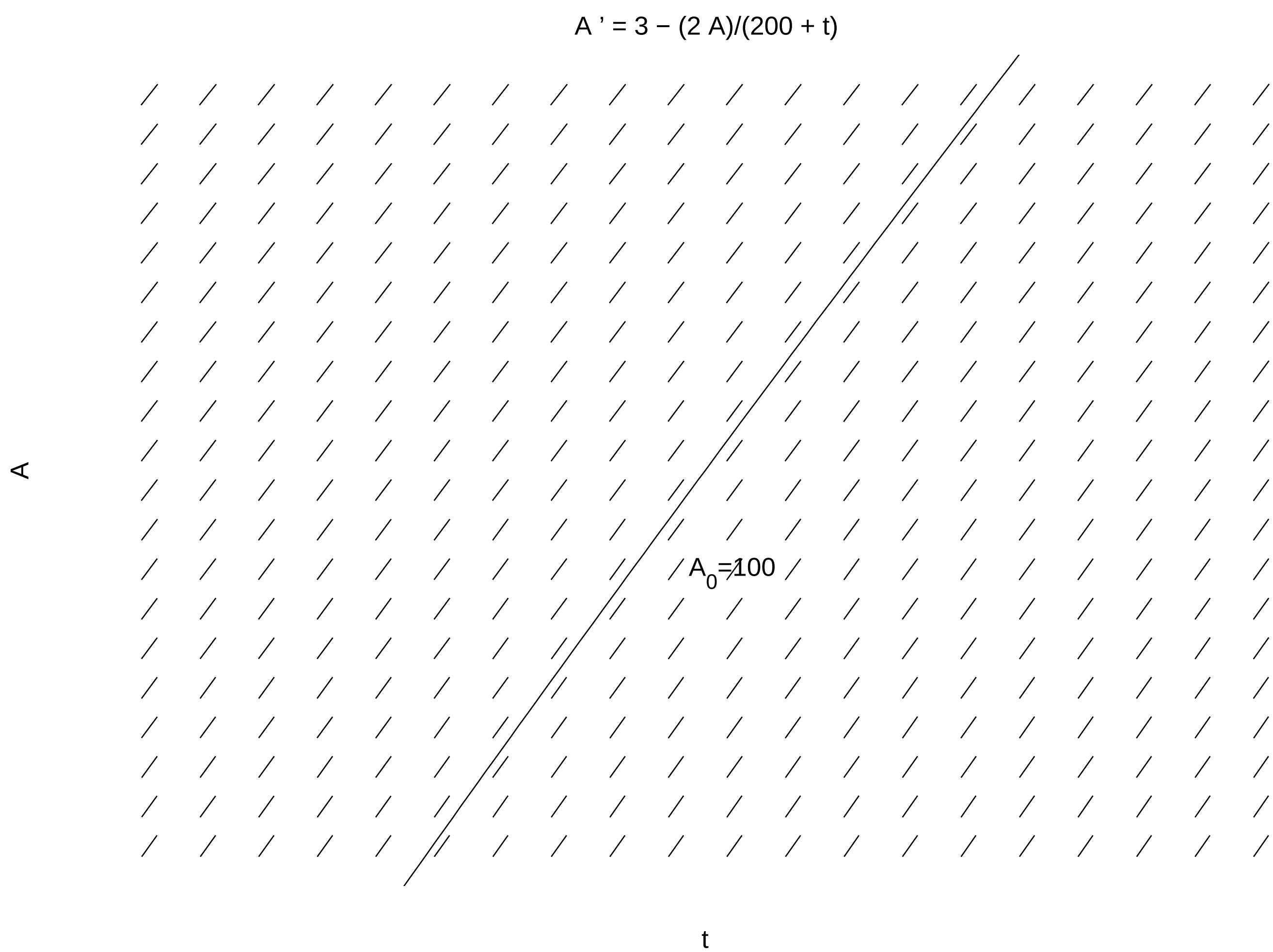
$$A(t) = (200+t)^{-2}\int 3(200+t)^2 dt = \frac{3}{(200+t)^{-2}}\int (200+t)^2 dt$$

$$= \frac{3}{(200+t)^{-2}}\left(\frac{1}{3}(200+t)^3 + C\right)$$

so

$$A(t) = t + 200 + \frac{K}{(200+t)^2} \qquad whereK = 4 \times 10^6$$

At $t = 300$, $A(t) = 340$.


A ' = 3 − (2 A)/(200 + t)
A
A0=100
t