§2.1

#3 Draw a direction field for the equation \( y' + y = e^{-t} + 1, \ y(0) = y_0 \)

![Direction field diagram](image)

We can solve the ODE using an integrating factor. If we set \( \mu(t) = e^t \) and multiply both sides of the equation by \( \mu \) we get:

\[
y'e^t + ye^t = (te^{-t} + 1)e^{-t} = t + e^t
\]

thus the new equation is:

\[
y'e^t + ye^t = (ye^t)' = t + e^t
\]

So, after integrating we get:

\[
ye^t = \frac{t^2}{2} + e^t + C
\]

Then we solve for \( y(t) \) to get

\[
y(t) = \left[ \frac{t^2}{2} + (1 - y_0) \right] e^{-t} + 1
\]

since \( C = 1 - y_0 \).

#13 Solve the IVP \( y' - y = 2te^{2t}, \ y(0) = 1 \) using an integrating factor. Again let \( \mu(t) \) be the integrating factor and choose \( \mu(t) = e^{-t} \). Multiplying the equation by \( \mu \) then integrating yields:

\[
ye^{-t} = \int (2te^{2t})e^{-t}dt = 2 \int te^t dt
\]
but
\[ \int te^t \, dt = te^t - e^t + C \]
so we have
\[ ye^{-t} = 2te^t - 2e^t + K \]
where \( K = 2C \). Solving for \( y(t) \) gives:
\[ y(t) = 2te^{2t} - 2e^{2t} + Ke^t \]
Then, using the initial condition to solve for \( K \) (\( K = 0 \)) we get:
\[ y(t) = 2te^{2t} - 2e^{2t} \]

**#29:** Consider the IVP \( y' - \frac{3}{2}y = 3t + 2e^t \), \( y(0) = y_0 \). Find the value of \( y_0 \) that separates solutions that grow positively as \( t \to \infty \) from those that grow negatively. Again we use an integrating factor to solve the IVP in terms of \( y_0 \) then take limits. The integrating factor is \( \mu(t) = e^{-\frac{3}{2}t} \). Then if we let \( g(t) = 3t + 2e^t \) and using the formula:
\[ y(t) = \frac{1}{\mu(t)} \int g(t)\mu(t)dt \]
we get:
\[ y(t) = Ke^{\frac{3}{2}t} - 4e^t - 2t - \frac{4}{3} \]
where \( K = y_0 + \frac{16}{3} \). Now if we let \( t \to \infty \) we get:
\[ \lim_{t \to \infty} y(t) = \begin{cases} +\infty & K > 0 \\ -\infty & K \leq 0 \end{cases} \]
Thus we have:
\[ \lim_{t \to \infty} y(t) = \begin{cases} +\infty & y_0 > \frac{16}{3} \\ -\infty & y_0 \leq \frac{16}{3} \end{cases} \]

**#35:** Consider the general 1st order, linear, ODE: \( y' + p(t)y = g(t) \).
\[ y' + p(t)y = g(t) \] \hspace{1cm} (1)
a) If \( g(t) \equiv 0 \) then we have \( y' + p(t)y = 0 \). This equation is easily solved using separation of variables. We get:
\[ y(t) = Ae^{-\int p(t)dt} \]
where \( A \) is a constant.
b) If \( g(t) \neq 0 \) we look for solutions of the form \( y(t) = A(t)e^{-\int p(t)dt} \) where the coefficient of the exponential, \( A(t) \), is no longer a constant, but a function of \( t \). If we differentiate we get:
\[ y'(t) = A'(t)e^{-\int p(t)dt} - A(t)p(t)e^{-\int p(t)dt} \]
subsituting this form into (1) we get:
\[ y' + py = (A' e^{-\int p\,dt} - A e^{-\int p\,dt}) + pA e^{-\int p\,dt} = g \]
or
\[ A'(t) = g(t)e^{\int p(t)dt} \]
c) Solving the above equation for \( A(t) \) we get:
\[ A(t) = \int g(t)e^{\int p(t)dt} \, dt \]
thus if we substitute this back into the assumed form of \( y(t) \) we have:
\[ y(t) = A(t)e^{-\int p(t)dt} = e^{-\int p(t)dt} \int g(t)e^{\int p(t)dt} \, dt \]
Now if we let \( \mu(t) = e^{\int p(t)dt} \) then the above equation becomes:
\[ y(t) = \frac{1}{\mu(t)} \int g(t)\mu(t) \, dt \]
which is the general solution to (1) obtained by using an integrating factor.

\[ \text{§2.2} \]
\[ \#5: \text{Solve the ODE } y' = \cos^2 x \cos^2 2y. \text{ We use separation of variables and get:} \]
\[ \frac{dy}{\cos^2 2y} = \cos^2 x \, dx \]
thus integrating both sides leaves:
\[ \frac{1}{2} \tan 2y = \int \cos^2 x \, dx = \frac{1}{2} \int (\cos 2x + 1) \, dx \]
(Note the use of the identity \( \cos^2 x = \frac{1}{2} (\cos 2x + 1) \). Remember it!) After integrating the left hand side we have:
\[ \tan 2y = \frac{1}{2} \sin 2x + 2x + K \]

\[ \#8: \text{Solve the ODE } \frac{dy}{dx} = \frac{x^2}{y^2+1}. \text{ Again we use separation of variables to get:} \]
\[ (y^2 + 1) \, dy = x^2 \, dx \]
Integrating yields:
\[ \frac{y^3}{3} + y - \frac{x^3}{3} = K \]
#12: Solve the IVP \( \frac{dr}{d\theta} = \frac{r^2}{\theta} \) \( r(1) = 2 \). Separating variable leaves:
\[
\frac{dr}{r^2} = \frac{d\theta}{\theta}
\]
Integrating and solving for \( r(\theta) \) gives:
\[
 r(\theta) = -\frac{1}{\ln \theta + K}
\]
Solving for the initial condition gives \( K = -\frac{1}{2} \), thus we have:
\[
 r(\theta) = -\frac{1}{\ln \theta - \frac{1}{2}}
\]

The interval of definition is \( 0 < \theta < \sqrt{e} \)

#23: Solve the IVP \( y' = 2y^2 + xy^2 \) \( y(0) = 1 \). The problem is non-linear, so the only technique we have learned that can be applied is Separation of Variables. Thus we separate:
\[
 \frac{dy}{y^2} = (2 + x)dx
\]
integrate:
\[- \frac{1}{y} = \frac{x^2}{2} + 2x + K\]
and finally solve for \(y(x)\) with the given initial conditions:
\[y(x) = -\frac{2}{x^2 + 4x - 2} \quad (-2 - \sqrt{6}) < x < (-2 + \sqrt{6})\]

Then, to find the minimum we differentiate and find the critical points to get \(x = -2\). This is in the domain of our solution, but we must check that it is a local-min. But \(y'(-3) < 0\) and \(y'(-1) > 0\), so \(x = -2\) is a local-min. Moreover, we have:
\[\lim_{x \to (-2 \pm \sqrt{6})} = +\infty\]
so \(x = -2\) is an absolute-min.

\[\frac{dA}{dt} = \text{Rate of change of the amount of chemical in the tank.}\]

Rate of Change = (Rate In) - (Rate Out)
\[\text{Rate In} = \frac{lb}{gal} \times \frac{gal}{min}\]

\section{§2.3}
\#4: To set up the differential equation let \(A(t) = \text{The amount (in poundss) of chemical in the tank at time } t\). Also in this problem the volume of solution is not fixed, so let \(V(t) = \text{The volume (in gallons) of solution in the tank at time } t\). Then we have the following:
Rate \ Out = \frac{A(t) \ lb}{V(t) \ gal} \times 2\frac{gal}{min}

The term \ \frac{\ A(t)}{V(t)} \ is \ the \ concentration \ of \ the \ solution \ leaving \ the \ tank. \ But \ V(t) \ is \ given \ as \ V(t) = 200 + t, \ so:\n
Rate \ Out = \frac{A(t) \ lb}{200 + t \ gal} \times 2\frac{gal}{min}

thus \ we \ have \ the \ following \ differential \ equation:

\frac{dA}{dt} = \left(1 \frac{lb}{gal} \times 3\frac{gal}{min}\right) - \left(\frac{A(t) \ lb}{200 + t \ gal} \times 2\frac{gal}{min}\right) = 3 - \frac{2A}{200 + t}

so \ the \ IVP \ we \ must \ solve \ is:

\frac{dA}{dt} + \frac{2A}{200 + t} = 3 \quad A(0) = 100

To solve \ this \ we \ use \ and \ integrating \ factor \ \mu(t) = e^{\int\frac{2}{200+t}dt} = (200 + t)^2. \ Thus \ if \ we \ take \ g(t) = 3 \ the \ general \ solution \ of \ the \ above \ equation \ is:

\begin{align*}
A(t) &= (200 + t)^{-2} \int 3(200 + t)^2dt \\
&= \frac{3}{(200 + t)^{-2}} \left( \frac{1}{3} (200 + t)^3 + C \right)
\end{align*}

so

A(t) = t + 200 + \frac{K}{(200 + t)^2} \quad \text{where} \ K = 4 \times 10^6

At \ t = 300, \ A(t) = 340.
A' = 3 - (2 A)/(200 + t)

A_0 = 100