§2.4

#1:Determine and interval in which a solution to the IVP is guaranteed to exist without solving the IVP.

$$(t-3)y' + \ln(t)y = 2t; \qquad y(1) = 2$$

First we put the ODE into standard form:

$$y' + \frac{\ln(t)}{(t-3)}y = \frac{2t}{(t-3)}$$

then we see that $p(t) = \frac{\ln(t)}{(t-3)}$, while $g(t) = \frac{2t}{(t-3)}$. Now, $p(t) = \frac{\ln(t)}{(t-3)}$ is continuous on the intervals $I_{p1} = (0,3)$ and $I_{p2} = (3,+\infty)$ while $g(t) = \frac{2t}{(t-3)}$ is continuous on the intervals $I_{g1} = (-\infty,3)$ and $I_{g2} = (3,+\infty)$. Thus we see that both $p(t)$ and $g(t)$ are continuous on I_{p1} and $I_{p2} = I_{g2}$. However, the initial condition is at $t = 1, y(t) = 2$; $t = 1$ is in the interval $I_{p1} = (0,3)$, thus we are guranteed a solution on the interval $I_{p1} = (0,3)$.

#4:Determine and interval in which a solution to the IVP is guaranteed to exist without solving the IVP.

$$(4-t^2)y' + 2ty = 3t^2; \qquad y(-3) = 1$$

First we put the ODE into standard form:

$$y' + \frac{2t}{(4-t^2)}y = \frac{3t^2}{(4-t^2)}$$

This time we see that bot $p(t)$ and $g(t)$ are continuous on the same domains, i.e. $I_1 = (-\infty,-2), I_2 = (-2,2)$, and $I_3 = (2,-\infty)$. So, since $t = -3 \in I_1 = (-\infty,-2)$ we have a guaranteed solution on $I_1 = (-\infty,-2)$.

#5:Determine and interval in which a solution to the IVP is guaranteed to exist without solving the IVP.

$$(4-t^2)y' + 2ty = 3t^2; \qquad y(1) = -3$$

This problem is the same as #4 except that the initial condition has changed. So we note that $t = 1 \in I_2 = (-2,2)$ so a solution is guaranteed on the interval $I_2 = (-2,2)$.

#8:State the region where the hypotheses of Theorem 2.4.2 are valid for the following ODE.

$$y' = \sqrt{1-(t^2+y^2)}$$

In this case $f(t,y) = \sqrt{1-(t^2+y^2)}$ which is continuous throughout the unit-disc in the (t,y) plane. Furthermore:

$$\frac{\partial f}{\partial y} = -\frac{y}{\sqrt{1-(t^2+y^2)}}$$

so we see that $\frac{\partial f}{\partial y}$ is continuous everywhere in the unit-disc, except on the boundary. Thus the hypotheses of Theorem 2.4.2 are satisfied on $\{(t,y) : t^2 + y^2 < 1\}$

#9:State the region where the hypotheses of Theorem 2.4.2 are valid for the following ODE.

$$y' = \frac{\ln|ty|}{1 - t^2 - y^2}$$

Here we have

$$f(t,y) = \frac{\ln|ty|}{1 - t^2 - y^2}$$

and

$$\frac{\partial f}{\partial y} = \frac{1}{y(1 - t^2 - y^2)} - 2y\frac{\ln|ty|}{(1 - t^2 - y^2)^2}$$

Thus we see that these are continuous on the set $\{(t,y) : t \neq 0, y \neq 0 \textit{ and } (t^2 - y^2) \neq 1\}$

#15: Solve the following IVP and determine how the interval in which the solution exists depends on the initial value.

$$y' + y^3 = 0; \qquad y(0) = y_0$$

Separating variables and integrating we get:

$$y^2 = \frac{1}{2t + K}$$

Solving for the intial condition gives $K = \frac{1}{y_0^2}$. Substituting this back into the equation and simplifying we get:

$$y(t) = +\sqrt{\frac{y_0^2}{2y_0^2 t + 1}}$$

Since $y_0^2 > 0$ we see that the above is defined as long as $(2y_0^2 t + 1) > 0$. Thus our interval on which the solution exists is $t > -\frac{1}{2y_0^2}$.

#22:Explain why the existence of the two solutions $y_1(t) = 1 - t$ and $y_2(t) = -\frac{t^2}{4}$ to the IVP

$$y' = \frac{-t + (t^2 + 4y)^{\frac{1}{2}}}{2}; \qquad y(2) = -1$$

does not contradict Theorem 2.4.2.

Theorem 2.4.2 guarantees a unique solution for every point in the region $y > -\frac{t^2}{4}$. The point (-1,2) does not satisfy this property, so no unique solution is guaranteed. c)If we let $y = ct + c^2$ where c is some constant. Then we note that $y' = c$ furthermore:

$$\frac{-t + (t^2 + 4y)^{\frac{1}{2}}}{2} = \frac{-t + (t^2 + 4(ct + c^2))^{\frac{1}{2}}}{2}$$

and

$$\frac{-t+(t^2+4(ct+c^2))^{\frac{1}{2}}}{2}=\frac{-t+(t^2+4ct+4c^2)^{\frac{1}{2}}}{2}=\frac{-t+[(t+2c)^2]^{\frac{1}{2}}}{2}$$

thus when we take $\sqrt{(t+2c)^2}$ we require that $t \geq -2c$ so that we can only take the positive square root, to get:

$$c=y'=\frac{-t+(t^2+4y)^{\frac{1}{2}}}{2}=\frac{-t+[(t+2c)^2]^{\frac{1}{2}}}{2}=c$$

Obviously for $c=-1$ we get $y_1(t)=1-t$, and trying to solve for c such that $y_2(t)=-\frac{t^2}{4}=ct+c^2$ we get that $t=-2c$, thus there is no constant c such that $-\frac{t^2}{4}=ct+c^2$.

#24:Show that if $y=\phi(t)$ is a solution to $y'+p(t)y=0$ then $y_c(t)=c\phi(t)$ is also a solution.

pf:

Let $y_c(t)$ be as above. Then $y_c'(t)=[c\phi(t)]'=c\phi'(t)$. Substituting into the equation we get:

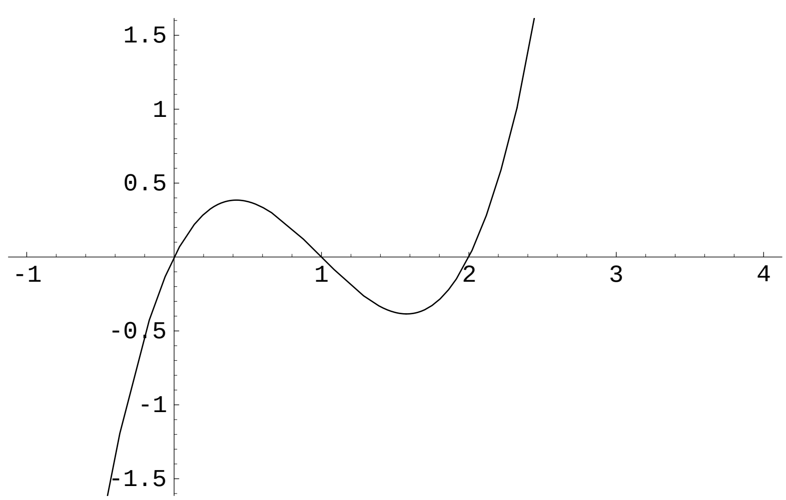
$$y_c'+p(t)y_c=c\phi'+p(t)c\phi=c(\phi'+p(t)\phi)=c\cdot 0=0$$

Thus $y_c(t)=c\phi(t)$ is a solution.

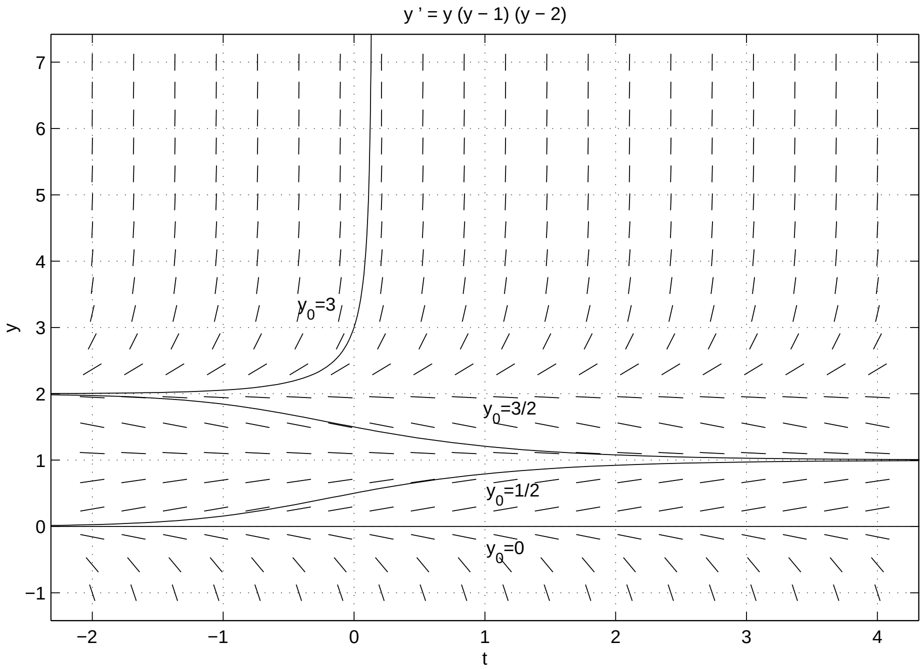
§2.5

#3:Show the graph of $f(y)$ versus y and classify the critical points.

$$y'=y(y-1)(y-2); \qquad y_0 \geq 0$$

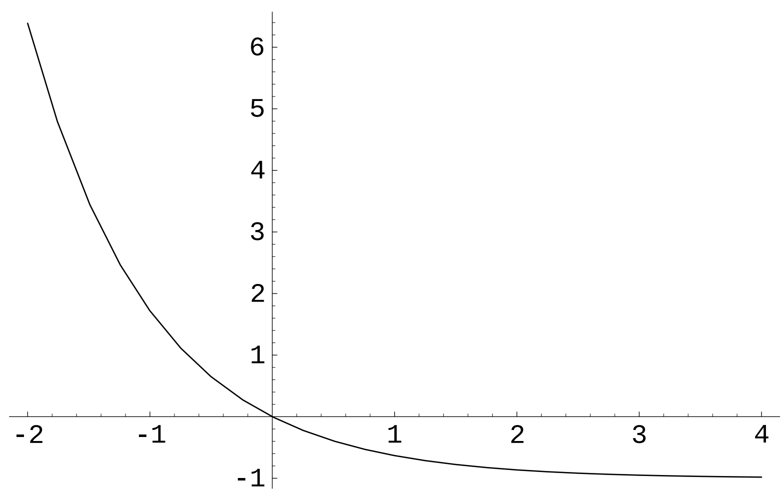


Thus we see that the equilibrium points are $y=0,\ y=1,\ y=2$ and from the image below we see that $y=0$, and $y=2$ are unstable, while $y=1$ is stable.

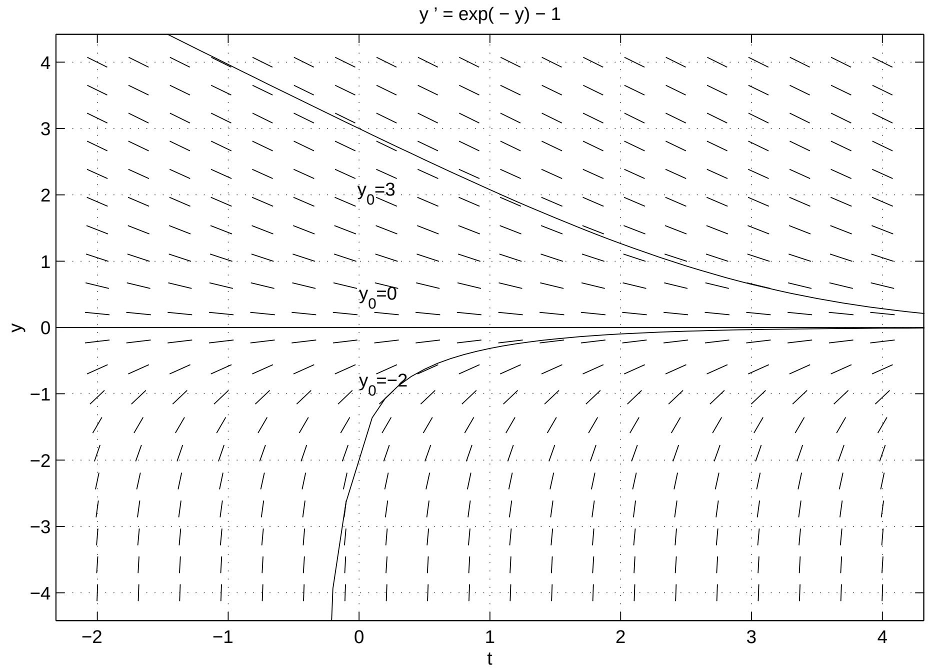


#5:Show the graph of $f(y)$ versus y and classify the critical points.

$$y' = e^{-y} - 1; \qquad -\infty < y_0, +\infty$$



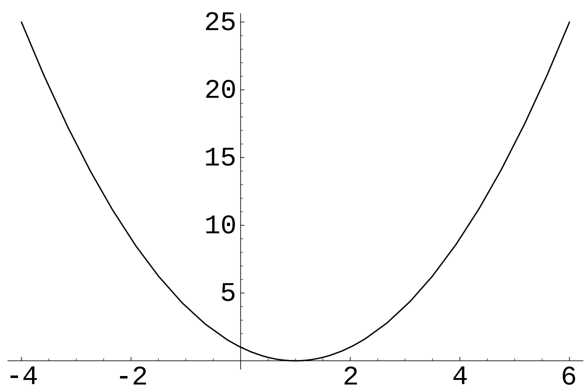
Thus we see that the only equilibrium point is $y = 0$, and from the following image we deduce that it is a stable point.



#7:Consider the ODE:

$$y' = k(1-y)^2; \qquad k > 0$$

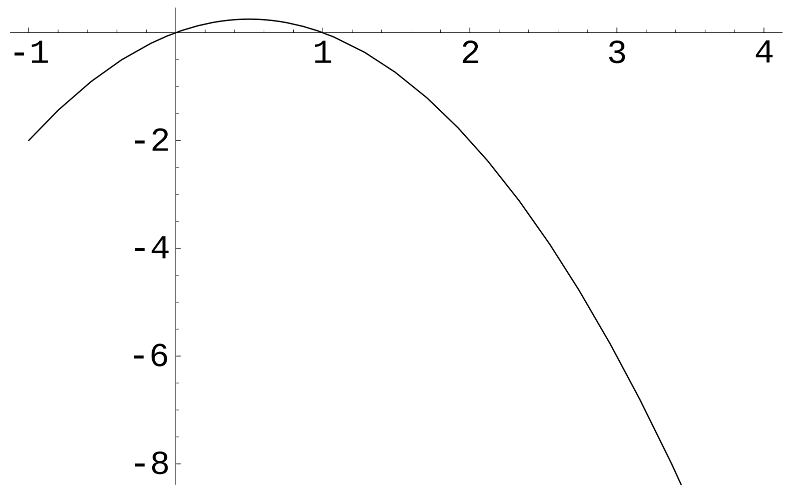
Notice that $y = 1$ is an equilibrium point, but $y' > 0$ for all $y \neq 1$, as can be seen by the following graph:



#22:Consider the following ODE which is an approximate model for the rate of growth of an infected population:

$$\frac{dy}{dt} = \alpha y(1-y); \qquad y(0) = y_0, \ \alpha > 0$$

The equilibrium points are $y = 0$ and $y = 1$. As can be seen from the following graph $y = 0$ is unstable while $y = 1$ is stable.



To solve this equation we separate variables:

$$\frac{dy}{y(1-y)} = \alpha dt$$

then we must integrate. The left hand side requires that we use partial-fractions, that is we assume that:

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}$$

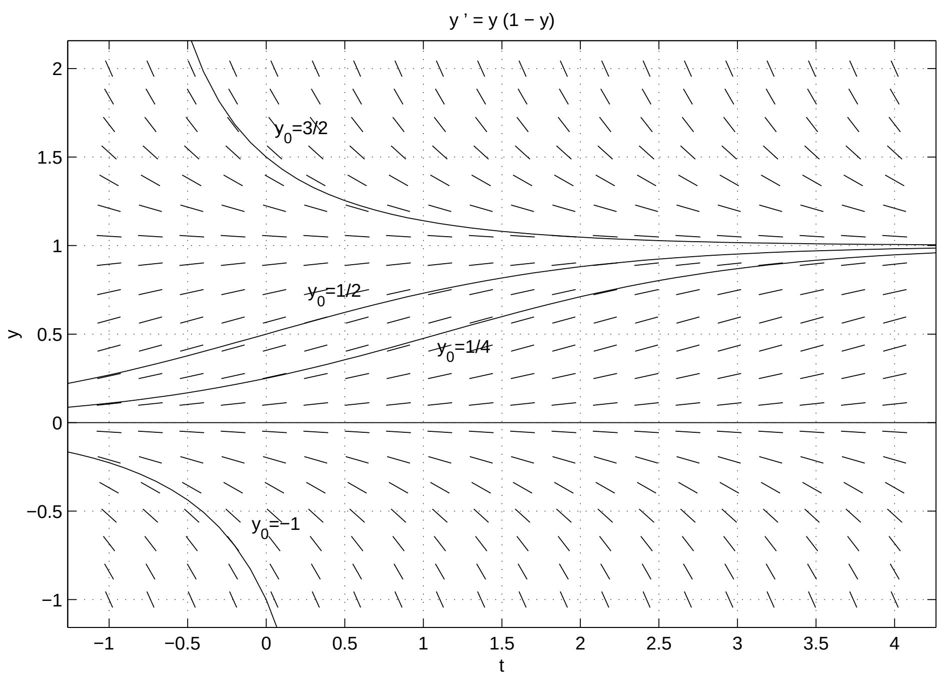
and then solve for A and B. Doing this we get:

$$\ln\left(\frac{y}{1-y}\right) = \alpha t + c$$

Solving for $y(t)$ with the IC $y(0) = y_0$ we get:

$$y(t) = \frac{y_0 e^{\alpha t}}{[1 + y_0(1 - e^{\alpha t})]}$$

Notice that $\lim_{t\to\infty} y(t) = 1$. A few solutions are plotted in the figure below. Note that here the only solutions with a "real" interpretation are those corresponding to $0 < y_0 < 1$ since $y(t)$ represents the proportion of the population infected at time t.


y ' = y (1 − y)
y
t
y0=3/2
y0=1/2
y0=1/4
y0=−1
2
1.5
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1
1.5
2
2.5
3
3.5
4