§2.7

#1:For the following differential equation give approximations of $y(t)$ for $t = .1, 0.2, 0.3, 0.4$ using stepsizes of $h = 0.1, 0.05, and 0.025$ using Euler's method.

$$y' = 3 + t - y \qquad y(0) = 1$$

Euler's method applied to the IVP gives:

$$y_{n+1} = y_n + hf_n$$

where $y' = f(t, y)$

$$y_{n+1} = y_n + h(3 + t_n - y_n)$$

but, $t_n = nh$, so

$$y_{n+1} = (1 - h)y_n + h(3 + nh)$$

with $y_0 = 1$. For the given values of t we get:

	$h = 0.1$	$h = 0.05$	$h = 0.025$
$t = 0.1$	1.2	1.1975	1.19631
$t = 0.2$	1.39	1.38549	1.38335
$t = 0.3$	1.571	1.56491	1.562
$t = .4$	1.7439	1.73658	1.73308

#15:For the IVP

$$y' = \frac{3t^2}{3y^2 - 4} \qquad y(1) = 0$$

find estimates for $t = 1.2, 1.4, 1.6, 1.8$ using step sizes of $h = 0.1, and 0.05$.

	$h = 0.1$	$h = 0.05$
$t = 1.2$	-0.16613	-0.174652
$t = 1.4$	-0.41087	-0.434238
$t = 1.6$	-0.80466	-0.88914
$t = 1.8$	4.1586	-3.0981

#20:This question is concerned with convergence of Euler's method. Consider the IVP

$$y' = 1 - t + y, \qquad y(t_0) = y_0$$

(a) Using an integrating factor we get that

$$y(t) = (y_0 - t_0)e^{t-t_0} + t$$

(b) However if we use the Euler formula for esimating the values of $y(t)$ we get:

$$y_k = y_{k-1} + f_{k-1}h$$

$$= y_{k-1} + (1 - t_{k-1} + y_{k-1})h$$
$$= (1+h)y_{k-1} + h - ht_{k-1}$$

(c) To show that $y_n = (1+h)^n(y_0 - t_0) + t_n \ \forall n$ we use induction.
for $n = 1$ we have
$$y_1 = (1+h)y_0 + h - ht_0$$
by the Euler formula. Now we add an subtract t_0 to get
$$y_1 = (1+h)y_0 + h - ht_0 = (1+h)y_0 - t_0 - ht_0 + h + ht_0$$
$$= (1+h)y_0 - (1+h)t_0 + (h + t_0)$$
$$= (1+h)(y_0 - t_0) + t_1$$
since $t_n = h + t_{n-1}$. Now, assume that the formula holds for $n = K$. Consider when $n = K+1$. By part (b) we have
$$y_{K+1} = (1+h)y_k + h - ht_k$$
Subtituting in for y_K we get
$$y_{K+1} = (1+h)\left[(1+h)^K(y_0 - t_0) + t_K\right] + h - ht_K$$
Distributing terms leaves
$$y_{K+1} = (1+h)^{K+1}(y_0 - t_0) + t_K + h + ht_K - ht_K$$
which gives
$$y_{K+1} = (1+h)^{K+1}(y_0 - t_0) + t_{K+1}$$
(d) Now if we set $h = \frac{(t-t_0)}{n}$ then $t_n = t \ \forall n$ so we have:
$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}(1+h)^n(y_0 - t_0) + t_n$$
$$= \lim_{n\to\infty}\left(1 + \frac{(t-t_0)}{n}\right)^n (y_0 - t_0) + t = e^{t-t_0}(y_0 - t_0) + t$$
which is the solution obtained using the integrating factor.

§2.8

#1:Convert the IVP $\frac{dy}{dt} = t^2 + y^2, \qquad y(1) = 2$ to an equivalent one with the initial condition at the origin.

Solution: We make a change of variable $\begin{matrix} t & \longrightarrow & t' \\ y & \longrightarrow & y' \end{matrix}$ such that $\begin{matrix} t = 1 & \longrightarrow & t' = 0 \\ y = 2 & \longrightarrow & y' = 0 \end{matrix}$. So we have:
$$t' = t - 1 \longrightarrow dt' = dt$$
$$y' = y - 2 \longrightarrow dy' = dy$$
Thus
$$\frac{dy}{dt} = \frac{dy'}{dt'} = (t'+1)^2 + (y'+2)^2$$

So the new IVP is:
$$\frac{dy'}{dt'} = (t'+1)^2 + (y'+2)^2 \qquad y(0) = 0$$

#4:Use Picard Iterates with $\phi_0 = 0$ to find approximate solutions for

$$y' = -y - 1 \qquad y(0) = 0$$

Solution Using the formula $\phi_n(t) = \int_0^t \phi_{n-1}(s)ds$ we get: $\phi_1 = -t$, $\phi_2 = \frac{t^2}{2} - t$, $\phi_3 = -\frac{t^3}{6} + \frac{t^2}{2} - t$, $\phi_4 = \frac{t^4}{24} - \frac{t^3}{6} + \frac{t^2}{2} - t$. So we see that

$$\phi_n(t) = \sum_{k=1}^{n} (-1)^k \frac{t^k}{k!}$$

If we solve the ODE using separation of variables we get $y(t) = e^{-t} + 1$. So we see that the Picard Iterates generate the Taylor Series for $y(t)$ which converges for all t.