§3.1 #1:Find the general solution of

$$y'' + 2y' - 3y = 0$$

Solution We will guess that the solution has the form $y = e^{rt}$. Thus, if we differentiate twice we get:

$$r^2e^{rt} + 2rert - 3e^{rt} = 0 \longrightarrow r^2 + 2r - 3 = 0$$

since $e^{rt} \neq 0 \ \ \forall t$. This implies

$$r^2 + 2r - 3 = 0 \longrightarrow (r-1)(r+2) = 0$$

thus we have the following:

$$y(t) = Ae^t + Be^{-2t}$$

as the general solution.

#6:Find the general solution of

$$4y'' - 9y = 0$$

We calculate the characteristic equation to be

$$4r^2 - 9 = 0$$

Thus the general solution becomes

$$y(t) = Ae^{\frac{3}{2}t} + Be^{-\frac{3}{2}t}$$

#15:Find the solution to the IVP

$$y'' + 8y' - 9y = 0 \qquad y(1) = 1, \ y'(1) = 0$$

The characteristic equation of the above is:

$$r^2 + 8r - 9 = 0$$

Thus the roots are $r = 1, -9$. Thus the general solution to the equation is :

$$y(t) = Ae^t + Be^{-9t}$$

Forcing this to satisfy the IC we get:

$$y(t) = \frac{9}{10}e^{t-1} + \frac{e^{9-9t}}{10}$$

#18:Find a differential equation whose solution is

$$y(t) = c_1 e^{-\frac{t}{2}} + c_2 e^{-2t}$$

Solution: If a 2^{nd} order ODE had the solution above then the roots of the characteristic equation would have roots $r = -\frac{1}{2}, -2$ Thus the characteristic equation becomes

$$(r + \frac{1}{2})(r + 2) = 0$$

Thus we have the folowing ODE:

$$2y'' + 3y' + 2y = 0$$

#23: Find the value of α for which the solutions to the following ODE go to zero; remain unbaounded.

$$y'' - (2\alpha - 1)r + \alpha(\alpha - 1)y = 0$$

Thus the roots of the characteristic equation are $r = \alpha, (\alpha - 1)$, so the general solution of the ODE becomes

$$y(t) = Ae^{\alpha t} + Be^{(\alpha-1)t}$$

Thus as long as $\alpha < 0$ then the solurions go to zero and if $\alpha > 0$ then the solution remains unboundend as $t \to \infty$.

§3.2

#1:Find the Wronskian of $y_1 = e^{2t}$ and $y_2 = e^{-\frac{3}{2}t}$.

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

Plugging y_1 and y_2 into the above we get:

$$W(y_1, y_2) = -\frac{7}{2} e^{\frac{t}{2}}$$

#2:Find the Wronskian of $\cos t$ and $\sin t$.
By above $W(\cos t, \sin t) = \cos^2 t + \sin^2 t = 1$.

#4:Find the Wronskian of x and xe^x.
By above $W(x, xe^x) = (xe^x + x^2 e^x) - xe^x = x^2 e^x$.

#11: Find the longest interval for a unique, twice differentiable solution to the IVP

$$(x-3)y'' + xy' + (\ln|x|)y = 0, \qquad y(1) = 0, \quad y'(1) = 1$$

Solution
By Theorem 3.2.1 the ODE $y'' + p(t)y' + q(t)y = g(t)$ has a unique soltuion on any interval where p, q, g are continuous. Thus we put the above IVP into the proper form:

$$y'' + \frac{x}{x-3}y' + \frac{\ln|x|}{x-3}y = 0$$

Then we have:

1. $p(t) = \frac{x}{x-3}$ which is continuous on $(-\infty, 3) \cup (3, +\infty)$.
2. $q(t) = \frac{\ln|x|}{x-3}$ which is continuous on $(-\infty, 0) \cup (0, 3) \cup (3, +\infty)$.

But $t_0 = 1$ thus the longest interval on which a unique twice differentiable solution is guaranteed to exist is $I = (0, 3)$.

#16:Can $\sin t^2$ be a solution to $y'' + p(t)y' + q(t)y = 0$ on an interval containig $t = 0$?
Solution
If $\sin t^2$ is a solution to the ODE then the equation holds for all t, particularly at $t = 0$. However

$$\sin'' t^2 + p(t)\sin' t^2 + q(t)\sin t^2|_{t=0} = 2 \neq 0$$

Thus $\sin t^2$ can not be a solution to the ODE on any interval containg $t = 0$.

#22:Find a fundamental set of solutions u_1 and u_2 to the ODE

$$y'' + 4y' + 3y = 0$$

with initial point $t_0 = 1$ such that we have:

$$\left.\begin{array}{ccc} u_1(t_0) = 1 & & u_2(t_0) = 0 \\ & ; & \\ u_1'(t_0) = 0 & & u_2'(t_0) = 1 \end{array}\right\} \quad t_0 = 1$$

Solution
The characteristic equation of the ODE is:

$$r^2 + 4r + 3 = 0 \longrightarrow (r+3)(r+1) = 0 \Rightarrow r = -1, -3$$

Thus we get a fundamental set of solutions:

$$\begin{cases} y_1 = e^{-t} \\ y_2 = e^{-3t} \end{cases}$$

However, y_1 and y_2 do not satisfy the initial conditions we are looking for, but since they form a fundamental set we can use them to find the desired u_1 and u_2. We proceed as follows: Let

$$\begin{cases} u_1 = Ay_1 + By_2 \\ u_2 = Cy_1 + Dy_2 \end{cases}$$

Now solve for the initial conditions. We get the two linear systems:

$$\begin{bmatrix} e^{-1} & e^{-3} \\ -e^{-1} & -3e^{-3} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad ; \quad \begin{bmatrix} e^{-1} & e^{-3} \\ -e^{-1} & -3e^{-3} \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

To solve these we calculate the inverse of the matrix:

$$\begin{bmatrix} e^{-1} & e^{-3} \\ -e^{-1} & -3e^{-3} \end{bmatrix}^{-1} = -\frac{1}{2}e^4 \begin{bmatrix} -3e^{-3} & -e^{-3} \\ e^{-1} & e^{-1} \end{bmatrix}$$

Thus we get:

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \frac{3}{2}e \\ -\frac{1}{2}e^3 \end{bmatrix} \quad ; \quad \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e \\ -\frac{1}{2}e^3 \end{bmatrix}$$

So that for u_1 and u_2 we get:

$$\begin{cases} u_1(t) = \frac{3}{2}e^{1-t} - \frac{1}{2}e^{3-3t} \\ \\ u_2(t) = \frac{1}{2}e^{1-t} - \frac{1}{2}e^{3-3t} \end{cases}$$

<u>#25</u>:Determine if $y_1 = x$ and $y_2 = xe^x$ form a fundamental set of solutions for

$$x^2y'' - x(x+2)y' + (x+2)y = 0$$

<u>Solution</u>
$W(x, xe^x) = x^2e^x$, so if we evaluate $W(1)$ we get $e \neq 0$. Thus $y_1 = x$ and $y_2 = xe^x$ form a fundamental set of solutions.

§3.3

<u>#2</u> Determine if $f = \cos 3\theta$ and $g = \cos^3\theta - 3\cos\theta$ are linearly independent or linearly dependent.
<u>Solution</u>
$W(f,g)(\frac{\pi}{6} = -\frac{27\sqrt{3}}{8} \neq 0$. Thus f and g are linearly independent.

<u>#9</u>Suppose that two functions have $W(y_1, y_2)(t) = t\sin^2 t$. Are the linearly independent of linearly dependent?
<u>Solution</u>
$W(\frac{\pi}{2}) = \frac{\pi}{2}\sin^2\frac{\pi}{2} = \frac{\pi}{2} \neq 0$. Thus by theorem 3.3.1 the functions are linearly independent.

#10 $W(t) = t^2 - 4$. To determine if y_1 and y_2 are linearly independent we evaluate $W(0) = -4 \neq 0$.Thus by theorem 3.3.1 the functions are linearly independent.