§3.3

#14:(a) To see that any vector in the plane can be written as a linear combination of $\hat{\imath}+\hat{\jmath}$ and $\hat{\imath}-\hat{\jmath}$ we just need to show that for all a, b we can find c_1 and c_2 such that $c_1(\hat{\imath}+\hat{\jmath})+c_2(\hat{\imath}-\hat{\jmath}) = a\hat{\imath}+b\hat{\jmath}$. But this is clearly true since the equation can be written in matrix form as:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

and this clearly has a solution for all a, b since the matrix has det $= -2$.
(b) If we suppose that $\vec{x} = x_1\hat{\imath}+x_2\hat{\jmath}$ and $\vec{y} = y_1\hat{\imath}+y_2\hat{\jmath}$ are linearly independent then we can write any $\vec{z} = z_1\hat{\imath}+z_2\hat{\jmath}$ as $c_1\vec{x}+c_2\vec{y}$. This can be seen similarly to part (a). If we write the corresponding matrix system we get:

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

but $\vec{x}$ and $\vec{y}$ are linearly independent so $x_1y_2 - x_2y_1 \neq 0$ thus the system has a solution.

#16: Using Abel's Theorem we see that

$$W(y_1, y_2)(t) = c\exp\left[-\int p(t)dt\right]$$

In this case

$$p(t) = \frac{\sin t}{\cos t}$$

Thus

$$W(t) = \cos t$$

#19: Suppose that $p(t)$ is differentiable, also, suppose that y_1, y_2 are solution to the equation

$$[p(t)y']' + q(t)y = 0$$

Then to find the Wronskian we simplify the abpove equation and use Abel's Theorem. We get:

$$p(t)y'' + p'(t)y' + q(t)y = y'' + \frac{p'}{p}y' + \frac{q}{p}y = 0$$

Now by Abel's Theorem

$$W(t) = c\exp\left[-\int \frac{p'}{p}dt\right]$$

which gives

$$W(t) = \frac{c}{p(t)}$$

#24: Suppose that y_1 and y_2 are two solutions to some ODE such that $y_1(t_0) = y_2(t_0) = 0$ then we see that $W(t_0) = 0$. Thus be Theorem 3.3 y_1 and y_2 cannot be a fundamental set of solutions to the ODE since if they were then the Wronskian would be non-zero.

#28: Consider the functions $f(t) = t^2|t|$ and $g(t) = t^3$. On the interval $0 < t < 1$ $f = g$ so they are linearly dependent. Also, on $-1 < t < 0$ $f = -g$ so they are linearly dependent, but on $-1 < t < 1$ the functions are linearly independent since they have opposite sign on $-1 < t < 0$ and the same sign on $0 < t < 1$.

§3.4

#3: $e^{i\pi} = \cos\pi + i\sin\pi = -1$

#9:Consider the differential equation

$$y'' + 2y' - 8y = 0$$

To find the general solution we assume a solution of the form $y = e^{rt}$ and get the characteristic equation:

$$r^2 + 2r - 8 = 0$$

and get the roots $r = -4, 2$. Thus the general solution is

$$y = Ae^{-4t} + Be^{2t}$$

#12:Consider the ODE:

$$4y'' + 9y = 0$$

To find the general solution we assume a solution of the form $y = e^{rt}$ and get the characteristic equation:

$$4r^2 + 9 = 0$$

and get roots $r = \pm\frac{3}{2}i$. So for a solution we would get

$$y = Ae^{i\frac{3}{2}t} + Be^{-i\frac{3}{2}t}$$

but this would give complex solutions. Using Euler's formula we see that $\cos\frac{3}{2}t$ and $\sin\frac{3}{2}t$ are linear combinations of the above and that they are linearly independent. Moreover they are real valued, so we set the general solution to the original ODE to:

$$y = A\cos\frac{3}{2}t + B\sin\frac{3}{2}t$$

where A and B are real.

#19:Consider the IVP

$$y'' - 2y' + 5y = 0 \qquad y(0) = 1, y'(0) = 0$$

To solve the problem we look at the characteristic equation $r^2 - 2r + 5 = 0$ and get roots$r = 1 \pm 2i$. Thus the general solution to the homogeneous equation is

$$y = e^t(A\cos 2t + B\sin 2t)$$

Solving for the initial conditions we get

$$y = e^t(\cos 2t - \frac{1}{2}\sin 2t)$$

#28:Consider the ODE

$$y'' + y = 0$$

$\cos t$ and $\sin t$ are clearly solutions. Moreover $W(\cos t, \sin t) = 1$, thus they form a fundamental set of solutions to the ODE. This means that any solution to the ODE is a linear combination of $\cos t$ and $\sin t$. Observe that e^{it} is also a solution to the ODE, and hence is a linear combination of $\cos t$ and $\sin t$, or in other words

$$e^{it} = c_1 \cos t + c_2 \sin t$$

we can solve for c_1 and c_2 by evaluating at $t = 0$ then differentiating and evaluating at $t = 0$ to get $c_1 = 1$ and $c_2 = i$.

#29:Consider Euler's Formula:

$$e^{it} = \cos t + i\sin t$$

Replacing t wit $-t$ and using that $\cos t$ is an even function and that $\sin t$ is odd we get:

$$e^{-it} = \cos -t + i\sin -t = \cos t - i\sin t$$

Thus if we add the first and second we get $e^{it} + e^{-it} = 2\cos t$ or

$$\frac{e^{it} + e^{-it}}{2} = \cos t$$

similarly

$$\frac{e^{it} - e^{-it}}{2i} = \sin t$$

3.5

#2:Consider the ODE

$$9y'' + 6y' + y = 0$$

To find the general solution we get the characteristic equation $9r^2 + 6r + 1 = (3r + 1)^2 = 0$ which has the repeated root $r = -\frac{1}{3}$. Thus we only get one linearly independent solution from the characteristic equation $y_1 = e^{-\frac{1}{3}t}$. To solve this problem we hypothesize a solution of the form $y(t) = v(t)y_1(t)$. Doing this we get that $v(t)$ must satisfy the ODE $v'' = 0$. Thus $v = c_1 t + c_2$ so we get the generla solution to the original equation as

$$y = c_1 t e^{-\frac{1}{3}t} + c_2 e^{-\frac{1}{3}t}$$

#12: Solve the IVP

$$y'' - 6y' + 9y = 0 \qquad y(0) = 0,\ y'(0) = 2$$

As before we look at the characteristic equation to get $r^2 - 6r + 9 = (r-3)^2 = 0$ which has one repeated root $r = 3$. So the general solution to the homogeneous equation is

$$y = c_1 t e^{3t} + c_2 e^{3t}$$

Then satisfying the initial condtions we get:

$$y = 2te^{3t}$$

#20:Consider the ODE

$$y'' + 2ay' + a^2 y = 0$$

To find the general solution we consider the characteristic equation $r^2 + 2ar + a^2 = (r+a)^2 = 0$ which has one repeated root $r = -a$. By Abel's Theorem

$$W(y_1, y_2)(t) = \exp\left[-\int 2a dt\right] = e^{-2at}$$

Thus if we note also that $W(y_1, y_2)(t) = y_1 y_2' - y_1' y_2$ then we can substitute the above for $W(t)$ and take $y_1 = e^{-at}$. Then we solve the resulting 1^{st} order equation for y_2 to get $y_2 = te^{-at}$

#25:Consider the ODE

$$t^2 y'' + 3ty' + y = 0$$

with the solution $y_1 = t^{-1}$. We can use reduction o order to find the second solution. First we guess a solution of the form

$$y_2 = v(t)y_1(t) = vt^{-1}$$

Then we differentiate and substitute back in to get that $v(t)$ satisfies

$$v''t + v' = 0$$

Thus for y_2 we get

$$y_2 = \frac{\ln t}{t}$$