§3.3

#14: (a) To see that any vector in the plane can be written as a linear combination of \( \hat{i} + \hat{j} \) and \( \hat{i} - \hat{j} \) we just need to show that for all \( a,b \) we can find \( c_1 \) and \( c_2 \) such that

\[
c_1(\hat{i} + \hat{j}) + c_2(\hat{i} - \hat{j}) = a\hat{i} + b\hat{j}.
\]

But this is clearly true since the equation can be written in matrix form as:

\[
\begin{bmatrix}
  1 & 1 \\
  1 & -1
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix}
= \begin{bmatrix}
  a \\
  b
\end{bmatrix}
\]

and this clearly has a solution for all \( a,b \) since the matrix has \( \det = -2 \).

(b) If we suppose that \( \vec{x} = x_1\hat{i} + x_2\hat{j} \) and \( \vec{y} = y_1\hat{i} + y_2\hat{j} \) are linearly independent then we can write any \( \vec{z} = z_1\hat{i} + z_2\hat{j} \) as \( c_1\vec{x} + c_2\vec{y} \). This can be seen similarly to part (a). If we write the corresponding matrix system we get:

\[
\begin{bmatrix}
  x_1 & y_1 \\
  x_2 & y_2
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix}
= \begin{bmatrix}
  z_1 \\
  z_2
\end{bmatrix}
\]

but \( \vec{x} \) and \( \vec{y} \) are linearly independent so \( x_1y_2 - x_2y_1 \neq 0 \) thus the system has a solution.

#16: Using Abel’s Theorem we see that

\[
W(y_1, y_2)(t) = c \exp \left[ - \int p(t)dt \right]
\]

In this case

\[
p(t) = \frac{\sin t}{\cos t}
\]

Thus

\[
W(t) = \cos t
\]

#19: Suppose that \( p(t) \) is differentiable, also, suppose that \( y_1, y_2 \) are solution to the equation

\[
[p(t)y']' + q(t)y = 0
\]

Then to find the Wronskian we simplify the abpove equation and use Abel’s Theorem. We get:

\[
p(t)y'' + p'(t)y' + q(t)y = y'' + \frac{p'}{p}y' + \frac{q}{p}y = 0
\]

Now by Abel’s Theorem

\[
W(t) = c \exp \left[ - \int \frac{p'}{p}dt \right]
\]

which gives

\[
W(t) = \frac{c}{p(t)}
\]
#24: Suppose that $y_1$ and $y_2$ are two solutions to some ODE such that $y_1(t_0) = y_2(t_0) = 0$ then we see that $W(t_0) = 0$. Thus by Theorem 3.3 $y_1$ and $y_2$ cannot be a fundamental set of solutions to the ODE since if they were then the Wronskian would be non-zero.

#28: Consider the functions $f(t) = t^2|t|$ and $g(t) = t^3$. On the interval $0 < t < 1$ $f = g$ so they are linearly dependent. Also, on $-1 < t < 0$ $f = -g$ so they are linearly dependent, but on $-1 < t < 1$ the functions are linearly independent since they have opposite sign on $-1 < t < 0$ and the same sign on $0 < t < 1$.

§3.4

#3: $e^{i\pi} = \cos \pi + i \sin \pi = -1$

#9: Consider the differential equation

$$y'' + 2y' - 8y = 0$$

To find the general solution we assume a solution of the form $y = e^{rt}$ and get the characteristic equation:

$$r^2 + 2r - 8 = 0$$

and get the roots $r = -4, 2$. Thus the general solution is

$$y = Ae^{-4t} + Be^{2t}$$

#12: Consider the ODE:

$$4y'' + 9y = 0$$

To find the general solution we assume a solution of the form $y = e^{rt}$ and get the characteristic equation:

$$4r^2 + 9 = 0$$

and get roots $r = \pm \frac{3}{2}i$. So for a solution we would get

$$y = Ae^{\frac{3}{2}it} + Be^{-\frac{3}{2}it}$$

but this would give complex solutions. Using Euler’s formula we see that $\cos \frac{3}{2}t$ and $\sin \frac{3}{2}t$ are linear combinations of the above and that they are linearly independent. Moreover they are real valued, so we set the general solution to the original ODE to:

$$y = A \cos \frac{3}{2}t + B \sin \frac{3}{2}t$$

where $A$ and $B$ are real.

#19: Consider the IVP

$$y'' - 2y' + 5y = 0 \quad y(0) = 1, y'(0) = 0$$
To solve the problem we look at the characteristic equation \( r^2 - 2r + 5 = 0 \) and get roots \( r = 1 \pm 2i \). Thus the general solution to the homogeneous equation is

\[ y = e^t(A \cos 2t + B \sin 2t) \]

Solving for the initial conditions we get

\[ y = e^t(\cos 2t - \frac{1}{2} \sin 2t) \]

**#28**: Consider the ODE

\[ y'' + y = 0 \]

cos \( t \) and sin \( t \) are clearly solutions. Moreover \( W(\cos t, \sin t) = 1 \), thus they form a fundamental set of solutions to the ODE. This means that any solution to the ODE is a linear combination of \( \cos t \) and \( \sin t \). Observe that \( e^{it} \) is also a solution to the ODE, and hence is a linear combination of \( \cos t \) and \( \sin t \), or in other words

\[ e^{it} = c_1 \cos t + c_2 \sin t \]

we can solve for \( c_1 \) and \( c_2 \) by evaluating at \( t = 0 \) then differentiating and evaluating at \( t = 0 \) to get \( c_1 = 1 \) and \( c_2 = i \).

**#29**: Consider Euler’s Formula:

\[ e^{it} = \cos t + i \sin t \]

Replacing \( t \) with \(-t\) and using that \( \cos t \) is an even function and that \( \sin t \) is odd we get:

\[ e^{-it} = \cos -t + i \sin -t = \cos t - i \sin t \]

Thus if we add the first and second we get \( e^{it} + e^{-it} = 2 \cos t \) or

\[ \frac{e^{it} + e^{-it}}{2} = \cos t \]

similarly

\[ \frac{e^{it} - e^{-it}}{2i} = \sin t \]

### 3.5

**#2**: Consider the ODE

\[ 9y'' + 6y' + y = 0 \]

To find the general solution we get the characteristic equation \( 9r^2 + 6r + 1 = (3r + 1)^2 = 0 \) which has the repeated root \( r = -\frac{1}{3} \). Thus we only get one linearly independent solution from the characteristic equation \( y_1 = e^{-\frac{1}{3}t} \). To solve this problem we hypothesize a solution
of the form \( y(t) = v(t)y_1(t) \). Doing this we get that \( v(t) \) must satisfy the ODE \( v'' = 0 \). Thus \( v = c_1t + c_2 \) so we get the general solution to the original equation as

\[
y = c_1te^{-\frac{1}{3}t} + c_2e^{-\frac{1}{3}t}
\]

\#12: Solve the IVP

\[
y'' - 6y' + 9y = 0 \quad y(0) = 0, \quad y'(0) = 2
\]

As before we look at the characteristic equation to get \( r^2 - 6r + 9 = (r - 3)^2 = 0 \) which has one repeated root \( r = 3 \). So the general solution to the homogeneous equation is

\[
y = c_1te^{3t} + c_2e^{3t}
\]

Then satisfying the initial conditions we get:

\[
y = 2te^{3t}
\]

\#20: Consider the ODE

\[
y'' + 2ay' + a^2y = 0
\]

To find the general solution we consider the characteristic equation \( r^2 + 2ar + a^2 = (r + a)^2 = 0 \) which has one repeated root \( r = -a \). By Abel’s Theorem

\[
W(y_1, y_2)(t) = \exp \left[ -\int 2adt \right] = e^{-2at}
\]

Thus if we note also that \( W(y_1, y_2)(t) = y_1y_2' - y_1' y_2 \) then we can substitute the above for \( W(t) \) and take \( y_1 = e^{-at} \). Then we solve the resulting 1st order equation for \( y_2 \) to get \( y_2 = te^{-at} \).

\#25: Consider the ODE

\[
t^2y'' + 3ty' + y = 0
\]

with the solution \( y_1 = t^{-1} \). We can use reduction of order to find the second solution. First we guess a solution of the form

\[
y_2 = v(t)y_1(t) = vt^{-1}
\]

Then we differentiate and substitute back in to get that \( v(t) \) satisfies

\[
v''t + v' = 0
\]

Thus for \( y_2 \) we get

\[
y_2 = \frac{\ln t}{t}
\]