Math 22B: HW 7

Sec. 3.6: 1, 9, 10, 18

1 Sec. 3.6

3.6.1) Given

$$y'' - 2y' - 3y = 3e^{2t},$$

to find the general solution, we must

(a) find the general solution to the corresponding homogeneous equation

$$y'' - 2y' - 3y = 0$$

(b) find a particular solution to the given equation.

(a) To solve

$$y'' - 2y' - 3y = 0$$

we note that the associated characteristic equation is

$$r^2 - 2r - 3 = 0$$

$$(r-3)(r+1) = 0$$

$$r = 3 \text{ or } r = -1.$$

So the general solution to the homogenous equation is

$$y_h = c_1e^{3t} + c_2e^{-t}.$$

(b) To find a particular solution to the given equation, we assume that the solution is of the form

$$y = Ae^{2t}$$

$$y' = 2Ae^{2t}$$

$$y'' = 4Ae^{2t}.$$

Substituting these back into the given ODE we get

$$4Ae^{2t} - 2(2Ae^{2t}) - 3(Ae^{2t}) = 3e^{2t}$$

$$-3A = 3$$

$$A = -1.$$

So, a particular solution is

$$y_p = -e^{2t}.$$

Therefore, our general solution is

$$y = y_h + y_p = c_1e^{3t} + c_2e^{-t} - e^{2t}.$$

3.6.9) Given

$$u'' + \omega_0^2 u = \cos(\omega t)$$

with $\omega^2 \neq \omega_0^2$,

- the corresponding homogeneous equation is

$$u'' + \omega_0^2 u = 0$$

and the associated characteristic equation to this equation is

$$r^2 + \omega_0^2 = 0$$

$$r = \pm \omega_0 i.$$

So, the general solution to the homogenous equation is

$$u_h = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

- Since $\omega^2 \neq \omega_0^2$, we use as our particular solution

$$u = A\cos(\omega t) + B\sin(\omega t)$$

$$u' = -A\omega \sin(\omega t) + B\omega \cos(\omega t)$$
$$u'' = -A\omega^2 \cos(\omega t) - B\omega^2 \sin(\omega t).$$

Substituting these back into the given equation, we get

$$-A\omega^2 \cos(\omega t) - B\omega^2 \sin(\omega t) + \omega_0^2[A\cos(\omega t) + B\sin(\omega t)] = \cos(\omega t)$$

$$-A\cos(\omega t)(\omega_0^2 - \omega^2) + B\sin(\omega t)(\omega_0^2 - \omega^2) = \cos(\omega t)$$

So,

$$B = 0$$

and

$$-A(\omega_0^2 - \omega^2) = 1$$
$$A = -\frac{1}{\omega_0^2 - \omega^2},$$

which means that our particular solution is

$$u_p = -\frac{1}{\omega_0^2 - \omega^2}\cos(\omega t).$$

So, the general solution to the given nonhomogeneous equation is

$$u = u_h + u_p = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) - \frac{1}{\omega_0^2 - \omega^2}\cos(\omega t).$$

3.6.10) Given

$$u'' + \omega_0^2 u = \cos(\omega_0 t)$$

- Note that the corresponding homogenous equation is the same as in problem 3.6.9 above. So

$$u_h = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

- To find a particular solution, we first note that the nonhomogenous term in the given equation, $\cos(\omega_0 t)$, is a solution to the homogeneous equation (with $c_1 = 1$ and $c_2 = 0$ in the above u_h.) So, the form of the particular solution that we use is

$$u = At\cos(\omega_0 t) + Bt\sin(\omega_0 t)$$

$$u' = A\cos(\omega_0 t) - At\omega_0 \sin(\omega_0 t) + B\sin(\omega_0 t) + Bt\omega_0 \cos(\omega_0 t)$$

$$\begin{aligned} u'' &= -A\omega_0 \sin(\omega_0 t) - A\omega_0 \sin(\omega_0 t) - At\omega_0^2 \cos(\omega_0 t) \\ &+ B\omega_0 \cos(\omega_0 t) + B\omega_0 \cos(\omega_0 t) - Bt\omega_0^2 \sin(\omega_0 t) \\ &= -2A\omega_0 \sin(\omega_0 t) - At\omega_0^2 \cos(\omega_0 t) + 2B\omega_0 \cos(\omega_0 t) - Bt\omega_0^2 \sin(\omega_0 t) \end{aligned}$$

Substituting these back into the given equation, we get

$$\begin{aligned} &-2A\omega_0 \sin(\omega_0 t) - At\omega_0^2 \cos(\omega_0 t) + 2B\omega_0 \cos(\omega_0 t) - Bt\omega_0^2 \sin(\omega_0 t) \\ &+ \omega_0^2 [At\cos(\omega_0 t) + Bt\sin(\omega_0 t)] = \cos(\omega_0 t) \end{aligned}$$

or

$$-2A\omega_0 \sin(\omega_0 t) + [At\omega_0^2 + 2B\omega_0 + At\omega_0^2] \cos(\omega_0 t) = \cos(\omega_0 t)$$

Since there is no sine term on the right side of the equation,

$$-2A\omega_0 = 0$$

$$A = 0.$$

Also, since the coefficient in front of the cosine term on the right side of the equation is 1,

$$At\omega_0^2 + 2B\omega_0 + At\omega_0^2 = 1$$

and since $A = 0$,

$$B = \frac{1}{2\omega_0}.$$

So, a particular solution is

$$u_p = \frac{1}{2\omega_0} t \sin(\omega_0 t)$$

Therefore the general solution is

$$u = u_h + u_p = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{1}{2\omega_0} t\sin(\omega_0 t).$$

3.6.18) Given

$$y'' + 2y' + 5y = 4e^{-t}\cos(2t)$$

and $y(0) = 1$ and $y'(0) = 0$

- The corresponding homogeneous equation is

$$y'' + 2y' + 5y = 0$$

and its characteristic equation is

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(5)}}{2}$$

$$r = -1 \pm 2i.$$

So, the general solution to the homogeneous equation is

$$y_h = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t).$$

- Since the homogeneous solution has the form

$$y_h = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$$

we multiply by t, for the form of the particular solution. So we assume a solution of the form

$$y = Ate^{-t}\cos(2t) + Bte^{-t}\sin(2t)$$

$$\begin{aligned} y' &= Ae^{-t}\cos(2t) - Ate^{-t}\cos(2t) - 2Ate^{-t}\sin(2t) \\ &+ Be^{-t}\sin(2t) - Bte^{-t}\sin(2t) + 2Bte^{-t}\cos(2t) \\ &= [A - At + 2Bt]e^{-t}\cos(2t) + [B - Bt - 2At]e^{-t}\sin(2t) \end{aligned}$$

$$\begin{aligned} y'' &= [-A + 2B]e^{-t}\cos(2t) - [A - At + 2Bt]e^{-t}\cos(2t) - 2[A - At + 2Bt]e^{-t}\sin(2t) \\ &\quad + [-B - 2A]e^{-t}\sin(2t) - [B - Bt - 2At]e^{-t}\sin(2t) + 2[B - Bt - 2At]e^{-t}\cos(2t) \\ &= [-2A + 4B - 3At - 4Bt]e^{-t}\cos(2t) + [-4A - 2B + 4At - 3Bt]e^{-t}\sin(2t). \end{aligned}$$

So, substituting these into the given equation, we get

$$\begin{aligned} &[-2A + 4B - 3At - 4Bt]e^{-t}\cos(2t) + [-4A - 2B + 4At - 3Bt]e^{-t}\sin(2t) \\ &+ 2\{[A - At + 2Bt]e^{-t}\cos(2t) + [B - Bt - 2At]e^{-t}\sin(2t)\} \\ &+ 5(Ate^{-t}\cos(2t) + Bte^{-t}\sin(2t)) = 4e^{-t}\cos(2t) \end{aligned}$$

which simplifies to

$$4Be^{-t}\cos(2t) - 4Ae^{-t}\sin(2t) = 4e^{-t}\cos(2t).$$

So

$$A = 0 \text{ and } B = 1,$$

which gives us

$$y_p = te^{-t}\sin(2t).$$

Hence, the general solution is

$$y = y_h + y_p = c_1e^{-t}\cos(2t) + c_2e^{-t}\sin(2t) + te^{-t}\sin(2t).$$

and

$$\begin{aligned} y' &= -c_1e^{-t}\cos(2t) - 2c_1e^{-t}\sin(2t) \\ &\quad - c_2e^{-t}\sin(2t) + 2c_2e^{-t}\cos(2t) \\ &\quad + e^{-t}\sin(2t) - te^{-t}\sin(2t) + 2te^{-t}\cos(2t). \end{aligned}$$

Now, $y(0) = 1$ implies

$$c_1 = 1$$

and $y'(0) = 0$ implies

$$-c_1 + 2c_2 = 0$$
$$c_2 = \frac{1}{2}.$$

So, our solution is

$$y = e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t) + te^{-t}\sin(2t).$$