§3.7

Problem #1 We find a particular solution of the ODE

$$y'' - 5y' + 6y = 2e^t$$

using the method of variation of parameters and then verify the solution using the method of undetermined coefficients.

VOP First we solve the homogeneous equation using the characteristic equation $r^2 - 5r + 6 = 0$ which has roots $r = 3, 2$. Thus a fundamental set of solutions for the homogeneous equation is

$$y_1 = e^{3t} \quad and \quad y_2 = e^{2t}$$

Thus we assume that a particular solution has the form

$$y_p(t) = u_1(t)e^{3t} + u_2(t)e^{2t}$$

differentiating we get that

$$\begin{aligned} y_p' &= 3u_1e^{3t} + 2u_2e^{2t} \\ y_p'' &= 3(u_1' + 3u_1)e^{3t} + 2(u_2' + 2u_2)e^{2t} \end{aligned}$$

The form of y_p' comes from the constituent equation

$$u_1'e^{3t} + u_2'e^{2t} = 0 \qquad (1)$$

Plugging y_p and its derivatives back into the ODE we get a second equation

$$3u_1'e^{3t} + 2u_2'e^{2t} = 2e^t \qquad (2)$$

Putting equations (1) and (2) together we get the system

$$\begin{bmatrix} e^{3t} & e^{2t} \\ 3e^{3t} & 2e^{2t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^t \end{bmatrix}$$

Solving the system we obtain

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 2e^{-2t} \\ -2e^{-t} \end{bmatrix}$$

Thus after integrating we get:

$$u_1 = -e^{-2t} \quad and \quad u_2 = 2e^{-t}$$

Returning to the original form we get

$$y_p(t) = u_1(t)e^{3t} + u_2(t)e^{2t} = -e^{-2t}e^{3t} + 2e^{-t}e^{2t} = e^t$$

UC Now we can check the answer above by the following: Assume that $y_p = Ae^t$ where A is a constant. Then we get

$$Ae^t - 5Ae^t + 6Ae^t = 2e^t \iff 2A = 2$$

Thus $A = 1$ and $y_p = e^t$

Problem #5 We use Variation of Parameters to solve the ODE

$$y'' + y = \tan t \qquad 0 < t < \frac{\pi}{2}$$

$y_h(t)$ The characteristic equation is $r^2+1 = 0$ which has roots $r = \pm i$. Thus the general solution to the homogeneous equation is

$$y_h = A\cos t + B\sin t$$

VOP We assume that

$$y_p(t) = u_1(t)\cos t + u_2(t)\sin t$$

Then the derivatives of y_p have the form

$$\begin{aligned} y_p' &= -u_1\sin t + u_2\cos t \\ y_p'' &= (u_2' - u_1)\cos t - (u_1' + u_2)\sin t \end{aligned}$$

The form of y_p' comes from the constituent equation

$$u_1'\cos t + u_2'\sin t = 0 \qquad (1)$$

Plugging y_p and its derivatives back into the ODE we get another equation

$$-u_1'\sin t + u_2'\cos t = \tan t \qquad (2)$$

Putting equations (1) and (2) together we get the system

$$\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \tan t \end{bmatrix}$$

Which after solving yields

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} -\sec t + \cos t \\ \sin t \end{bmatrix}$$

$u_1(t)$ Now we have $u_1'(t) = -\sec t + \cos t$ thus

$$u_1(t) = \int (-\sec t + \cos t)dt = -\int \sec t dt + \int \cos t dt$$

The second integral is easy, but to do the first we write

$$\sec t = \frac{1}{\cos t} = \frac{1}{\cos t}\cdot\frac{\cos t}{\cos t} = \frac{\cos t}{1 - \sin^2 t}$$

So

$$\int \sec t dt = \int \left(\frac{\cos t}{1 - \sin^2 t} \right) dt$$

Now, let $u = \sin t$ and $du = \cos t dt$, then

$$\int \left(\frac{\cos t}{1 - \sin^2 t} \right) dt = \int \frac{du}{1 - u^2} = \frac{1}{2} \left[\int \frac{du}{1 + u} + \int \frac{du}{1 - u} \right] = \frac{1}{2} \ln \left| \frac{1 + u}{1 - u} \right|$$

So finally we get

$$\int \sec t dt = \int \left(\frac{\cos t}{1 - \sin^2 t} \right) dt = \frac{1}{2} \ln \left| \frac{1 + \sin t}{1 - \sin t} \right|$$

Thus

$$u_1(t) = \sin t - \frac{1}{2} \ln \left| \frac{1 + \sin t}{1 - \sin t} \right|$$

$\underline{u_2(t)}$ Thankfully, the expression for u_2' is much easier. We have $u_2' = \sin t$ thus

$$u_2(t) = -\cos t$$

$\underline{y_p(t)}$ Now we assumed that $y_p(t) = u_1(t) \cos t + u_2(t) \sin t$ Thus

$$y_p(t) = \left[\sin t - \frac{1}{2} \ln \left| \frac{1 + \sin t}{1 - \sin t} \right| \right] \cos t + [-\cos t] \sin t$$

So

$$y_p(t) = -\frac{1}{2} \ln \left| \frac{1 + \sin t}{1 - \sin t} \right| \cdot \cos t$$

$\underline{y(t)}$ Now for the general solution $y(t) = y_h(t) + y_p(t)$, thus

$$y(t) = A \cos t + B \sin t - \frac{1}{2} \ln \left| \frac{1 + \sin t}{1 - \sin t} \right| \cdot \cos t$$

Problem #16 Given that $y_1 = e^t$ and $y_2 = t$ form a fundamental set of solutions for $(1 - t)y'' + ty' - y = 0$ we find a particular solution for

$$(1 - t)y'' + ty' - y = 2(t - 1)^2 e^{-t}$$

Assume that

$$y_p(t) = u_1 e^t + u_2 t$$

Then the derivatives of y_p become

$$y_p' = u_1 e^t + u_2$$
$$y_p'' = (u_1 + u_1')e^t + u_2'$$

with the constituent equation

$$u_1' e^t + u_2' t = 0 \qquad (1)$$

Plugging y_p and its derivatives into the ODE we get

$$u_1'e^t + u_2' = 2(1-t)e^{-t} \qquad (2)$$

Putting (1) and (2) together we have

$$\begin{bmatrix} e^t & t \\ e^t & 1 \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2(1-t)e^{-t} \end{bmatrix}$$

Inverting gives

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} -2te^{-2t} \\ 2e^{-t} \end{bmatrix}$$

Thus, after integrating, we get

$$u_1(t) = te^{-2t} + \frac{1}{2}e^{-2t} \quad \textit{and} \quad u_2(t) = -2e^{-t}$$

Thus

$$y_p(t) = u_1e^t + u_2t = \left(te^{-2t} + \frac{1}{2}e^{-2t}\right)e^t + (-2e^{-t})t = \frac{1}{2}e^{-t} - te^{-t}$$

Problem #21 Let $y(t)$ be any solution to the IVP

$$L[y] = y'' + p(t)y' + q(t)y = g(t)\ ; \quad y(t_0) = y_0\ ;\ y'(t_0) = y_0'$$

By Theorem 3.2.1 there exists a unique function $u(t)$ which is a solution to the IVP

$$L[u] = u'' + p(t)u' + q(t)u = 0\ ; \quad u(t_0) = y_0\ ;\ u'(t_0) = y_0'$$

Then let $v(t) = y(t) - u(t)$. We have

$$L[v] = L[y-u] = L[y] - L[u] = g(t) - 0 = g(t)$$

and

$$v(t_0) = y(t_0) - u(t_0) - y_0 - y_0 = 0\ ;\ v'(t_0) = y'(t_0) - u'(t_0) = y_0' - y_0' = 0$$

Thus $y(t) = v(t) + u(t)$ is a solution to the original IVP.

§6.1

Problem #5 (a) Find $\mathcal{L}[t]$

$$\mathcal{L}[t] = \int_0^\infty te^{-st}dt = \left[-\frac{1}{s}te^{-st} - \frac{1}{s^2}e^{-st}\right]_0^\infty = \frac{1}{s^2}$$

(b) Find $\mathcal{L}[t^2]$

$$\mathcal{L}[t^2] = \int_0^\infty t^2e^{-st}dt = \left[-\frac{1}{s}t^2e^{-st} - \frac{2}{s^2}e^{-st} - \frac{2}{s^3}e^{-st}\right]_0^\infty = \frac{2}{s^3}$$

(c) Find $\mathcal{L}[t^n]$.

$$\mathcal{L}[t^n] = \int_0^\infty t^n e^{-st} dt = \frac{n!}{s^{n+1}}$$

The reason is that all the terms in the integral are of the form $t^k e^{-st}$ $k \neq 0$ except for the last term. The coefficient of the last term is $-\frac{n!}{s^{n+1}}$.

Problem #9 Find $\mathcal{L}[e^{at}\cosh bt]$

$$\mathcal{L}[e^{at}\cosh bt] = \mathcal{L}\left[e^{at}\left(\frac{e^{bt}+e^{-bt}}{2}\right)\right] = \frac{1}{2}(\mathcal{L}[e^{(a+b)t}] + \mathcal{L}[e^{(a-b)t}])$$

$$\frac{1}{2}(\mathcal{L}[e^{(a+b)t}] + \mathcal{L}[e^{(a-b)t}]) = \frac{1}{2}\left[\frac{1}{s-(a+b)} + \frac{1}{s-(a-b)}\right] = \frac{s-a}{(s-a)^2-b^2}$$

Thus

$$\mathcal{L}[e^{at}\cosh bt] = \frac{s-a}{(s-a)^2-b^2}$$

Problem #13 Find $\mathcal{L}[e^{at}\sin bt]$

$$\mathcal{L}[e^{at}\sin bt] = \frac{1}{2i}\left[\mathcal{L}[e^{(a+ib)t}] - \mathcal{L}[e^{(a-ib)t}]\right] = \frac{1}{2i}\left[\frac{1}{s-(a+ib)} - \frac{1}{s-(a-ib)}\right] = \frac{b}{(s-a)^2+b^2}$$

Thus

$$\mathcal{L}[e^{at}\sin bt] = \frac{b}{(s-a)^2+b^2}$$

Problem #18 Find $\mathcal{L}[t^n e^{at}]$

$$\mathcal{L}[t^n e^{at}] = \int_0^\infty t^n e^{at} e^{-st} dt = \int_0^\infty t^n e^{(a-s)t} dt$$

Let $S = s - a$. Then

$$\int_0^\infty t^n e^{(a-s)t} dt = \int_0^\infty t^n e^{-St} dt = \frac{n!}{S^{n+1}} = \frac{n!}{(s-a)^{n+1}}$$

Thus

$$\mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}}$$

§6.2

Problem #11 Solve the IVP

$$y'' - y' - 6y = 0 \ ; \quad y(0) = 1 \ ; \ y'(0) = -1$$

using a Laplace transform.

$$\mathcal{L}[y'' - y' - 6y] = (s^2 Y(s) - sy(0) - y'(0)) - (sY(s) - y(0)) - 6Y(s) = 0$$

$$\implies Y(s) = \frac{s-2}{s^2-s-6} = \frac{1}{5}\cdot\frac{1}{s-3} + \frac{4}{5}\cdot\frac{1}{s+2}$$

Thus

$$y(t) = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}$$

Problem #14 Solve the IVP

$$y'' - 4y' + 4y = 0 \ ; \qquad y(0) = 1 \ ; \ y'(0) = 1$$

using a Laplace transform.

$$\mathcal{L}[y'' - 4y' + 4y] = (s^2Y(s) - sy(0) - y'(0)) - 4(sY(s) - y(0)) + 4Y(s) = 0$$

$$\implies Y(s) = \frac{s-3}{(s-2)^2} = \frac{s-2}{(s-2)^2} - \frac{1}{(s-2)^2}$$

$$\implies Y(s) = \frac{1}{s-2} - \frac{1}{(s-2)^2}$$

Thus

$$y(t) = e^{2t} - te^{2t} = (1-t)e^{2t}$$

Problem #16 Solve the IVP

$$y'' + 2y' + 5y = 0 \ ; \qquad y(0) = 2 \ ; \ y'(0) = -1$$

using a Laplace transform.

$$\mathcal{L}[y'' + 2y' + 5y] = (s^2Y - sy(0) - y'(0)) + 2(sY - y(0)) + 5Y = 0$$

$$\implies Y(s) = \frac{2s+3}{s^2+2s+5} = \frac{(2s+2)+1}{(s^2+2s+1)+4} = 2 \cdot \frac{s+1}{(s+1)^2+4} + \frac{1}{2} \cdot \frac{2}{(s+1)^2+4}$$

Thus

$$y(t) = 2e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t$$

Problem #18 Solve the IVP

$$y^{iv} - y = 0 \ ; \qquad y(0) = 1 \ ; \ y'(0) = 0 \ ; \ y''(0) = 1 \ ; \ y'''(0) = 0$$

using a Laplace transform.

$$\mathcal{L}[y^{iv} - y] = (s^4Y - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)) - Y = 0$$

$$\implies Y(s) = \frac{s^3+s}{s^4-1} = \frac{s}{s^2-1}$$

Thus

$$y(t) = \cosh t$$

Problem #21 Solve the IVP

$$y'' - 2y + 2y = \cos t \ ; \qquad y(0) = 1 \ ; \ y'(0) = 0$$

using a Laplace transform.

$$\mathcal{L}[y'' - 2y' + 2y] = (s^2Y - sy(0) - y'(0)) - 2(sY - y(0)) + 2Y = \frac{s}{s^2+1} = \mathcal{L}[\cos t]$$

$$\implies Y(s) = \underbrace{\left(\frac{s-2}{s^2-2s+2}\right)}_{(1)} + \underbrace{\left(\frac{s}{(s^2+1)\cdot(s^2-2s+2)}\right)}_{(2)}$$

We can deal with the two parts (1) and (2) seperately since $\mathcal{L}^{-1}$ is a linear operator. Furthermore (1) corresponds to the homogeneous equation and (2) to a particular solution to the given ODE.

(1) We have

$$Y_h(s) = \frac{s-2}{s^2-2s+2} = \frac{s-2}{(s-1)^2+1} = \frac{s-1}{(s-1)^2+1} - \frac{1}{(s-1)^2+1}$$

Thus

$$y_h(t) = e^t \cos t - e^t \sin t$$

(2) Here we have

$$Y_p(s) = \frac{s}{(s^2+1)\cdot(s^2-2s+2)}$$

Now we assume that

$$\frac{s}{(s^2+1)\cdot(s^2-2s+2)} = \frac{As+B}{s^2-2s+2} + \frac{Cs+D}{s^2+1}$$

Solving this we get $A = -\frac{1}{5}; B = \frac{4}{5}; C = \frac{1}{5}; D = -\frac{2}{5}$. Thus

$$Y_p(s) = \frac{1}{5}\left[\frac{-s+4}{s^2-2s+2} + \frac{s-2}{s^2+1}\right] = \frac{1}{5}\left[-\frac{(s-1)-3}{(s-1)^2+1} + \frac{s}{s^2+1} - 2\cdot\frac{1}{s^2+1}\right]$$

And finally

$$Y_p(t) = -\frac{1}{5}\cdot\frac{(s-1)}{(s-1)^2+1} + \frac{3}{5}\cdot\frac{1}{(s-1)^2+1} + \frac{1}{5}\cdot\frac{s}{s^2+1} - \frac{2}{5}\cdot\frac{1}{s^2+1}$$

Hence

$$y_p(t) = -\frac{1}{5}e^t\cos t + \frac{3}{5}e^t \sin t + \frac{1}{5}\cos t - \frac{2}{5}\sin t$$

Finally if we put (1) and (2) together we get

$$y(t) = y_h(t) + y_p(t) = \frac{4}{5}e^t\cos t - \frac{2}{5}e^t\sin t + \frac{1}{5}\cos t - \frac{2}{5}\sin t$$

Proble #23 Solve the IVP

$$y'' + 2y' + y = 4e^{-t}\ ; \quad y(0) = 2\ ;\ y'(0) = -1$$

using a Laplace transform.

$$\mathcal{L}[y''+2y'+y] = (s^2Y - sy(0) - y'(0)) + 2(sY - y(0)) + Y = \frac{4}{s+1} = \mathcal{L}[4e^{-4t}]$$

$$\implies Y(s) = \frac{2s+3}{s^2+2s+1} + \frac{4}{(s+1)(s^2+2s+1)} = \frac{2(s+1)+1}{(s+1)^2} + \frac{4}{(s+1)^3}$$

Thus

$$Y(s) = s \cdot \frac{1}{s+1} + \frac{1}{(s+1)^2} + 2 \cdot \frac{2}{(s+1)^3}$$

And finally

$$y(t) = 2e^{-t} + te^{-t} + 2t^2e^{-t}$$