§7.1

Problem #4 Write the following ODE as a system of first order equations.

$$u^{iv} - u = 0$$

Let $x_1 = u,\ x_2 = u',\ x_3 = u'',\ x_4 = u'''$. Then the corresponding system is:

$$u^{iv} - u = 0 \Longleftrightarrow \begin{cases} x_1' = & x_2 \\ x_2' = & x_3 \\ x_3' = & x_4 \\ x_4' = & x_1 \end{cases}$$

Problem #7 Consider the system $x_1' = -2x_1 + x_2$ and $x_2' = x_1 - 2x_2$.

(a) Put into one euation and then find a general solution for the system.

$$x_1' = -2x_1 + x_2 \longrightarrow \begin{array}{rl} x_2 = & x_1' + 2x_1 \\ x_2' = & x_1'' + 2x_1' \end{array}$$

Thus we have

$$x_2' = x_1 - 2x_2 \longrightarrow x_1'' + 2x_1' = x_1 - 2(x_1' + 2x_1)$$

Thus we get

$$x_1'' + 4x_1' + 3x_1 = 0$$

So, we solve this equation and substitute back into the above expression for x_2 to get:

$$\begin{array}{rl} x_1 = & c_1e^{-t} + c_2e^{-3t} \\ x_2 = & c_1e^{-t} - c_2e^{-3t} \end{array}$$

(b) Solve the IVP if we set $x_1(0) = 2$ and $x_2(0) = 3$. If we set $t = 0$ we get the system of linear equations:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Solving this system we get the solution:

$$\begin{array}{rl} x_1 = & \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ x_2 = & \frac{5}{2}e^{-t} + \frac{1}{2}e^{-3t} \end{array}$$

Problem #11 Consider the IVP

$$\begin{array}{rl} x_1' = & \frac{5}{4}x_1 + \frac{3}{4}x_2 \\ x_2' = & \frac{3}{4}x_1 + \frac{5}{4}x_1' \end{array} \quad x_1(0) = -2,\ x_2(0) = 1$$

To solve this problem we solve for x_2 in terms of x_1 and x_1' to get:

$$\begin{array}{rl} x_2 = & \frac{4}{3}x_1' - \frac{5}{3}x_1 \\ x_2' = & \frac{4}{3}x_1'' - \frac{5}{3}x_1' \end{array}$$

Then substituting back into the equation for x_1' and simplifying, we get

$$16x_1'' - 40x_1' + 16x_1 = 0$$

which has characteristic equation

$$16r^2 - 40r + 16 = (16r - 8)(r - 2) = 0 \Longrightarrow r = \frac{1}{2}, 2$$

Thus for x_1 we get

$$x_1(t) = c_1 e^{\frac{1}{2}t} + c_2 e^{2t}$$

and for x_2 we have

$$x_2(t) = -c_1 e^{\frac{1}{2}t} + c_2 e^{2t}$$

Then solving for the IC we get $c_1 = -\frac{3}{2}$ and $c_2 = -\frac{1}{2}$ so the final solution is:

$$\begin{aligned} x_1 &= -\tfrac{3}{2}e^{\frac{1}{2}t} - \tfrac{1}{2}e^{2t} \\ x_1 &= \tfrac{3}{2}e^{\frac{1}{2}t} - \tfrac{1}{2}e^{2t} \end{aligned}$$

Problem #15 Consider the system $\vec{x}' = P\vec{x}$ where the matrix P is given by

$$P = \begin{matrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{matrix}$$

Show that if $\vec{x}_1$ and $\vec{x}_2$ are both solution then so is $c_1\vec{x}_1 + c_2\vec{x}_2$.

proof: Let $\vec{v} = c_1\vec{x}_1 + c_2\vec{x}_2$. Then:

$$P\vec{v} = P(c_1\vec{x}_1 + c_2\vec{x}_2) = c_1 P\vec{x}_1 + c_2 P\vec{x}_2 = c_1\vec{x}_1' + c_2\vec{x}_2' = (c_1\vec{x}_1 + c_2\vec{x}_2)'$$

§7.2

Problem #25 Verify that the given matrix satisfies the differential equation.

$$\Psi' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \Psi \qquad \Psi(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$$

solution:

$$\Psi' = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}' = \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \Psi = \begin{pmatrix} (1 \cdot e^{-3t} - 4 \cdot e^{-3t}) & (1 \cdot e^{2t} + 1 \cdot e^{2t}) \\ (4 \cdot e^{-3t} - 2 \cdot (-4)e^{-3t}) & (4 \cdot e^{2t} - 2 \cdot e^{2t}) \end{pmatrix} = \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix}$$

Thus the given matrix satisfies the ODE.

§7.3

Problem #13 Determine if the vectors

$$\vec{x}_1 = \begin{pmatrix} 2\sin t \\ \sin t \end{pmatrix} \quad \text{and} \quad \vec{x}_2 = \begin{pmatrix} \sin t \\ 2\sin t \end{pmatrix}$$

are linearly independent.

solution: Consider the equation $c_1\vec{x}_1 + c_2\vec{x}_2 = \vec{0}$ in order to determine the linear independence or dependence of the given vectors we must find the solutions to this equation. But this equation is equivalent to:

$$\sin t \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

But the only solutions to the above are clearly $c_1 = c_2 = 0$ since $\sin t \neq 0$ for some t and the determinant of the matrix is 3. Thus the vectors are linearly independent.

Problem #18 Find the eigen-values and eigen-vectors for $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

solution: First we find the eigen-values by calculating the char. eqn. $det(\lambda I - A) = 0$. We get:

$$\lambda^2 - 2\lambda = \lambda(\lambda - 2) = 0 \longleftarrow \lambda = 0, 2$$

So solving for the eigen vectors we get:

$$\lambda = 0;\ \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad \lambda = 2;\ \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Problem #24 Find the eigen-vectors and eigen-values of $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

solution: First we calculate the characteristic equation

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = (\lambda^2 + 2\lambda + 1)(\lambda - 8) = (\lambda + 1)^2(\lambda - 8) = 0$$

Thus the roots are $\lambda = -1, 8$. Now we calculate the $Ker(\lambda I - A)$ for $\lambda = -1, 8$. Note that $\lambda = -1$ has multiplicity two, so it has two linearly independent eigen-vectors since A is symmetric matrix.

$$\lambda = -1;\ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix};\ \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \qquad \lambda = 8;\ \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

§7.4

Problem #4 Consider the ODE and equivalent system given below

$$y'' + p(t)y' + q(t)y = 0 \longleftrightarrow \begin{array}{rl} x_1' = & x_2 \\ x_2' = & -q(t)x_1 - p(t)x_2 \end{array}$$

Let $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ be a fundamental set of solutions to the system and $y^{(1)}$ and $y^{(2)}$ be a fundamental set for the equation. Then we know that $x_1^{(1)}$ and $x_1^{(2)}$ form a fundamental set of solutions to the equation, so we have that

$$\begin{array}{rl} y^{(1)} = & c^1x_1^1 + c^2x_1^2 \\ y^{(2)} = & c^3x_1^1 + c^4x_1^2 \end{array}$$

where all superscripts are indexes, not powers. Thus, if we calculate $W(y^{(1)}, y^{(2)})$ we get:

$$W(y^{(1)}, y^{(2)}) = \begin{vmatrix} y^{(1)} & y^{(2)} \\ y^{(1)\prime} & y^{(2)\prime} \end{vmatrix} = \begin{vmatrix} (c^1x_1^1 + c^2x_1^2) & (c^3x_1^1 + c^4x_1^2) \\ (c^1x_1^{1\prime} + c^2x_1^{2\prime}) & (c^3x_1^{1\prime} + c^4x_1^{2\prime}) \end{vmatrix}$$

But $x_1^{1\prime} = x_2^1$ and $x_1^{2\prime} = x_2^2$ thus we have:

$$\begin{vmatrix} (c^1x_1^1 + c^2x_1^2) & (c^3x_1^1 + c^4x_1^2) \\ (c^1x_1^{1\prime} + c^2x_1^{2\prime}) & (c^3x_1^{1\prime} + c^4x_1^{2\prime}) \end{vmatrix} = \begin{vmatrix} (c^1x_1^1 + c^2x_1^2) & (c^3x_1^1 + c^4x_1^2) \\ (c^1x_2^1 + c^2x_2^2) & (c^3x_2^1 + c^4x_2^2) \end{vmatrix}$$

Expanding this out and simplifying we get:

$$W(y^{(1)}, y^{(2)}) = (c^1x_1^1+c^2x_1^2)(c^3x_2^1+c^4x_2^2) - (c^1x_2^1+c^2x_2^2)(c^3x_1^1+c^4x_1^2) = (c^1c^4 - c^2c^3)W[\vec{x}^{(1)}, \vec{x}^{(2)}]$$

Problem #6 Let

$$\vec{x}^{(1)} = \begin{pmatrix} t \\ 1 \end{pmatrix} \quad \textit{and} \quad \vec{x}^{(2)} = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$$

(a): $W \begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix} = 2t^2 - t^2 = t^2$

(b): The vectors are linearly independent on $(-\infty, 0)$ and $(0, +\infty)$

(c): The coefficients of the system are not continuous at $t = 0$

Problem #8 Let $\vec{x}^1, \ldots, \vec{x}^m$ be solutions to $\vec{x}' = P(t)\vec{x}$ where all the coefficients of the matrix P are continuous for $\alpha < t < \beta$. Assume that $\vec{x}^1(t_0), \ldots, \vec{x}^m(t_0)$ are linearly independent. We show that $\vec{x}^1, \ldots, \vec{x}^m$ are linearly independent for $\alpha < t < \beta$.

proof: If $\vec{x}^1(t_0), \ldots, \vec{x}^m(t_0)$ are linearly independent for $t = t_0$ then $W[\vec{x}^1, \ldots, \vec{x}^m](t_0) \neq 0$. But we have that $t_0 \in (\alpha, \beta)$ and $P(t)$ is continuous for $t \in (\alpha, \beta)$ thus $W[\vec{x}^1, \ldots, \vec{x}^m](t) \neq 0$ for all $t \in (\alpha, \beta)$. Thus $\vec{x}^1, \ldots, \vec{x}^m$ are linearly independent on (α, β)

Problem #9 Let $\vec{x}^1, \ldots, \vec{x}^m$ be linearly independent solutions to $\vec{x}' = P(t)\vec{x}$ where all the coefficients of the matrix P are continuous for $\alpha < t < \beta$. Show that any solution $\vec{z}(t)$ can be written as

$$\vec{z}(t) = c_1\vec{x}^1(t) + \cdots + c_m\vec{x}^m(t)$$

proof: Let $\vec{z}(t)$ be a solution. Fix $t = t_0$. Then we have that $\vec{z}(t_0) = c_1\vec{x}^1(t_0) + \cdots + c_m\vec{x}^m(t_0)$ has a unique solution for all $t_0 \in (\alpha, \beta)$ since $\vec{x}^1, \ldots, \vec{x}^m$ are linearly independent. Moreover the $c_j's$ do not depend on t since $c_j(t)\vec{x}^j(t)$ is not a solution to the given system.

§7.5

Problem#9 Find the general solution of the ODE

$$\vec{x}' = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \vec{x}$$

solution: From before we have

$$\lambda = 0; \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \textit{and} \quad \lambda = 2; \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

so the general soution is:

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Problem #18 Solve the IVP

$$\vec{x}' = \begin{pmatrix} 0 & 0 & -1 \\ -2 & 0 & 0 \\ -1 & 0 & 4 \end{pmatrix} \vec{x}; \quad \vec{x}(0) = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix}$$

solution: Calculating the eigen-values and eigen-vectors we get:

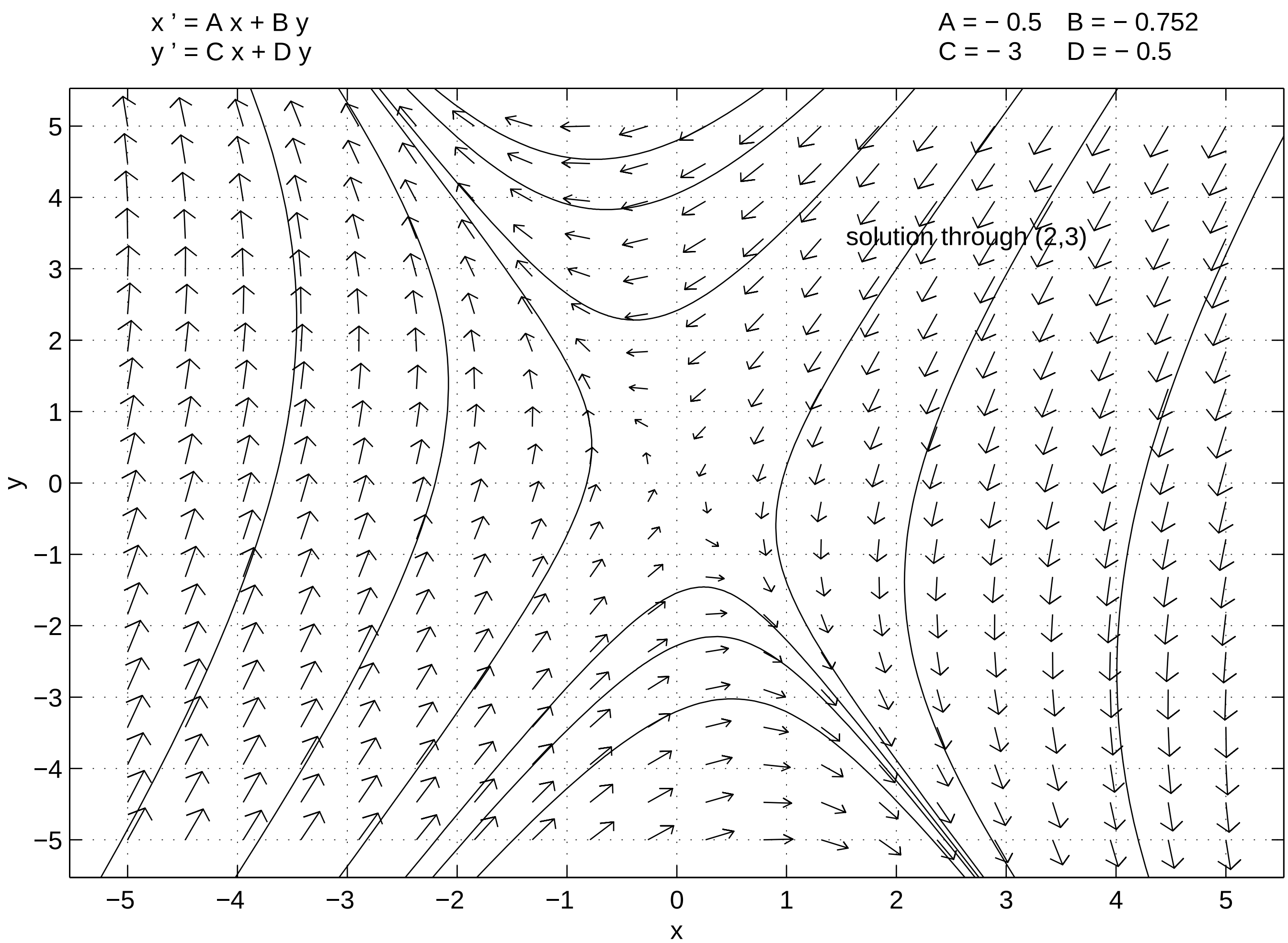
$$\lambda_1 = 1; \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \lambda_2 = -1; \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \lambda_3 = 4; \begin{pmatrix} -2 \\ -1 \\ 8 \end{pmatrix}$$

So for the general solution we get:

$$\vec{x}(t) = c_1 e^t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} -2 \\ -1 \\ 8 \end{pmatrix}$$

. Then solving for the IC we get

$$\vec{x}(t) = 6e^t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 3e^{-t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + e^{4t} \begin{pmatrix} -2 \\ -1 \\ 8 \end{pmatrix}$$


x ' = A x + B y
y ' = C x + D y
A = − 0.5 B = − 0.752
C = − 3 D = − 0.5
solution through (2,3)
x
y
−5
−4
−3
−2
−1
0
1
2
3
4
5