MATH 22B-001: Differential Equations
MidTerm Exam I, Wednesday January 31, 2001

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Total Score:____________________

Problem 1 (20 pts) Consider the following initial value problem (IVP).

$$\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{2}{t}y = t^2 e^t, \quad t \geq 1, \quad y(1) = 0.$$

(a) (5 pts) Is this differential equation linear or nonlinear?

(b) (5 pts) What method would you use to find a solution to this IVP?

(c) (10 pts) Solve this IVP.

Solution
(a)Linear
(b)Integrating factor or variation of parameters
(c)Multiply on both sides by integrating factor $\mu(t)$:

$$\mu y' - \frac{2}{t}\mu y = \mu t^2 e^t$$

Let

$$\mu y' - \frac{2}{t}\mu y = \mu y' + \mu' y$$

which gives

$$\mu' = -\frac{2}{t}\mu \longrightarrow \frac{\mu'}{\mu} = -\frac{2}{t} \longrightarrow \ln|\mu| = -2\ln|t|$$

so we pick

$$\mu(t) = \frac{1}{t^2}$$

Therefore we have

$$(\frac{1}{t^2}y)' = \frac{1}{t^2} \cdot t^2 e^t = e^t$$

Integration yields:

$$\frac{1}{t^2}y = e^t + C \longrightarrow y = t^2(e^t + C)$$

Now, solving for the initial condition we get

$$0 = y(1) = 1^2(e^1 + C) \longrightarrow C = -e$$

so

$$y(t) = t^2(e^t - e)$$

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Problem 2 (30 pts) Consider the following differential equation.

$$(1+t)\,\frac{\mathrm{d}y}{\mathrm{d}t} = 1 + y.$$

(a) (5 pts) Is this differential equation linear or nonlinear?

(b) (5 pts) What method would you use to find a solution of this differential equation?

(c) (10 pts) Solve this differential equation.

(d) (5 pts) Draw several integral curves by varying the integration constant c.

(e) (5 pts) All the integral curves pass through a common point on the ty-plane. What is the coordinate of this common point?

<u>Solution</u>
(a)Linear
(b)Separation of Variables
(c)Assume that $1+t \neq 0$ and $1+y \neq 0$ then we have:

$$\frac{dy}{1+y} = \frac{dt}{1+t}$$

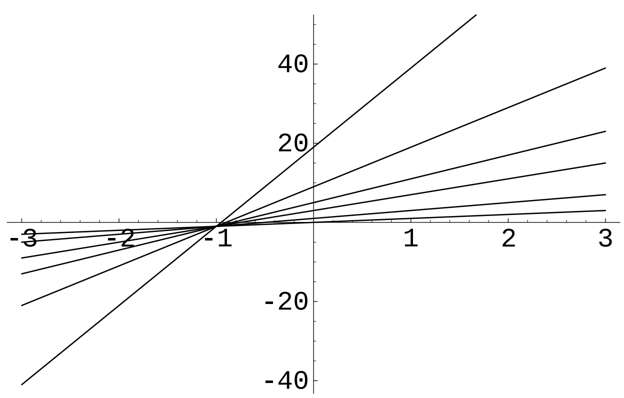
integrating gives:

$$\ln|1+y| = \ln|1+t| + C \longrightarrow |1+y| = e^C|1+t|$$

So, we have:

$$y(t) = K(t+1) - 1$$

where K is an arbitrary constant. (d)



(e)$(t,y) = (-1,-1)$ for all K.

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Problem 3 (30 pts) Consider the following initial value problem of *population dynamics*:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -y(y-\alpha)(y-\beta), \quad y(0) = y_0 > 0,$$

where $0 < \alpha < \beta < \infty$.

(a) (9 pts) What are the equilibrium solutions? You must also state whether each equilibrium solution is asymptotically stable or unstable.

(b) (8 pts) Draw several integral curves by varying y_0.

(c) (8 pts) Repeat (b) for the case $\alpha = \beta$.

(d) (5 pts) In order to avoid extinction, what condition y_0 must satisfy (regardless of $\alpha < \beta$ or $\alpha = \beta$) ?

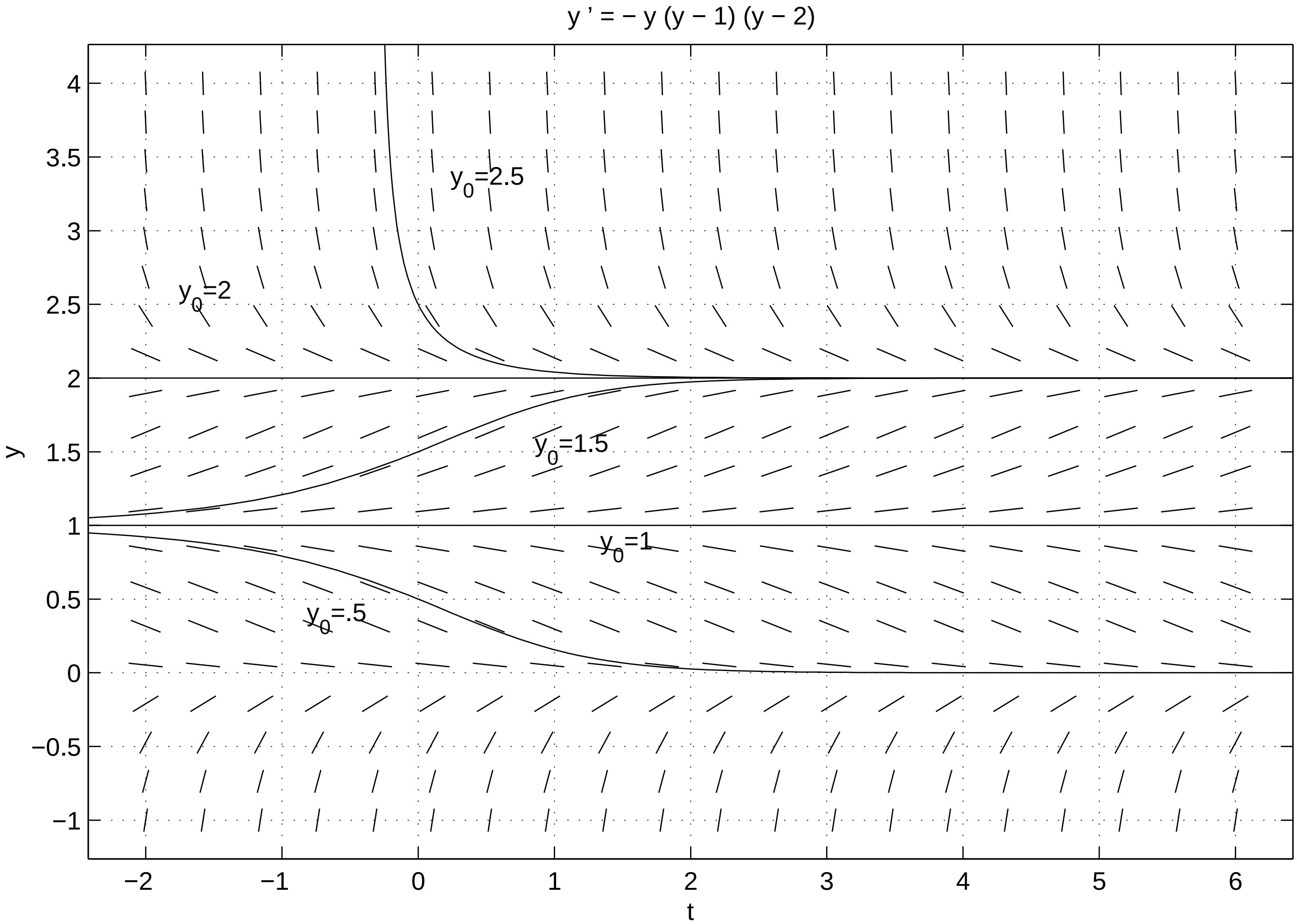
<u>Solution</u>
(a)The equilibrium solutions must satisfy

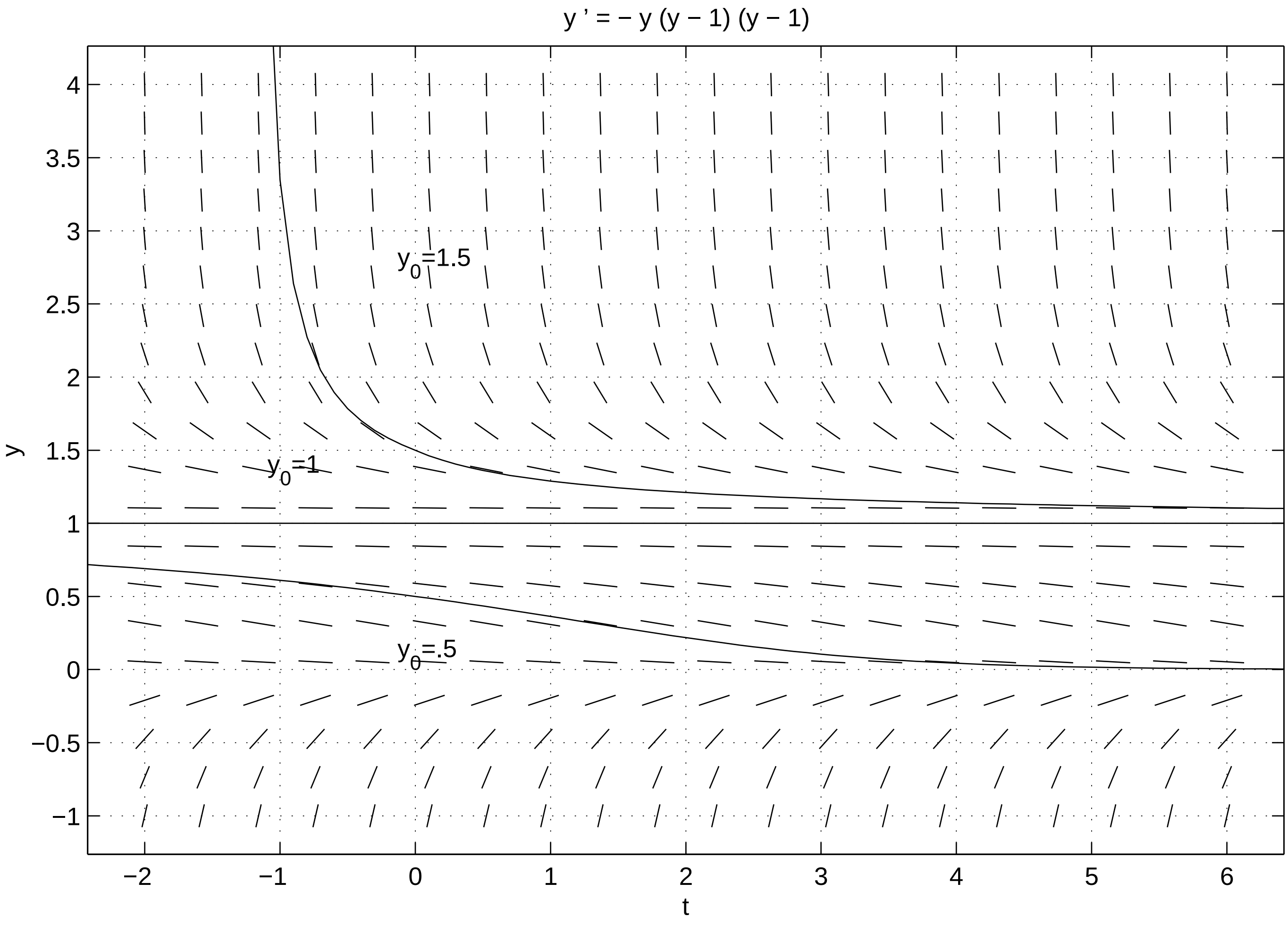
$$f(y) = -y(y-\alpha)(y-\beta) = 0$$

thus, we have $y = 0, \alpha, \beta$. To classify each as stable or unstable we note

$$f'(0) < 0;\ f'(\alpha) > 0;\ and\ f'(\beta) < 0$$

So $y = 0, \beta$ are assymptotically stable while $y = \alpha$ is unstable.
(b)In the figure below $\alpha = 1$ and $\beta = 2$.
(c)In the figure below $\alpha = \beta = 1$
(d)Regardless of $0 < \alpha < \beta$ or $0 < \alpha = \beta$, we need to have $y_0 \geq \alpha$. Note that $y_0 = \alpha$ is included.


y ' = − y (y − 1) (y − 2)
y
t
4
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−2
−1
0
1
2
3
4
5
6
y0=2.5
y0=2
y0=1.5
y0=1
y0=.5


y ' = − y (y − 1) (y − 1)
y
4
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−2
−1
0
1
2
3
4
5
6
t
y0=1.5
y0=1
y0=.5

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Select one of the following problems (either Problem 4-1 or 4-2), and solve it. You don't have to solve both of them, but if you want to and can solve both of them, you will get extra points.

Problem 4-1 (20 pts) Consider the following initial value problem (IVP)

$$\frac{\mathrm{d}y}{\mathrm{d}x} - (\tan x)\, y = 0, \quad y(\pi) = 1.$$

(a) (5 pts) Is this differential equation linear or nonlinear?

(b) (5 pts) What is the interval of x where the solution to this IVP exists?

(c) (10 pts) Solve this IVP. [Hint: $\frac{d}{dx} \ln|\cos x| = -\frac{\sin x}{\cos x}$.]

Solution
(a)Linear
(b)From the theorem of existence and uniqueness of the solution to an IVP, we have to have an interval I where $\tan x$ is conintuous and $x = \pi \in I$. To find the I we need we write

$$\tan x = \frac{\sin x}{\cos x}$$

Thus, $\tan x$ is continuos as long as $\cos x \neq 0$. But $\cos x = 0$ for $x = \ldots - \frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} \ldots$. So we see that we have a unique solution on $I = (\frac{\pi}{2}, \frac{3\pi}{2})$
(c)We have $y' - \tan x y = 0$. This is a separable equation. Thus we have:

$$\frac{dy}{y} = \tan x dx$$

Integrating gives:

$$ln|y| = -\ln|\cos x| + C \longrightarrow y = \frac{K}{\cos x}$$

where K is a constant. Now solving for the initial condition $y(\pi) = 1$ we get $K = -1$. Thus

$$y(t) = -\frac{1}{\cos x}$$

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Problem 4-2 (20 pts) Consdier the following IVP.

$$\frac{\mathrm{d}y}{\mathrm{d}t} - 2ty = 1, \quad 0 \le t \le 1, \quad y(0) = 1.$$

(a) (10 pts) Derive Euler's formula for this equation. Use h as a step size.

(b) (10 pts) Let $h = 0.5$ and compute y_0, y_1, and y_2 numerically by hand.

Solution

(a)$y' = 2ty + 1 = f(t, y)$ and $t_i = ih,\ i = 0, 1, 2, \ldots$

$$\begin{aligned} y_0 &= 1 \\ y_{i+1} &= y_i + hf(t_i, y_i) \\ &= y_i + h(2t_i y_i + 1) \\ &= y_i + h(2 \cdot ih \cdot y_i + 1) \\ &= (1 + 2h^2 i) y_i + h \end{aligned}$$

(b)

$$\begin{aligned} y_0 &= 1 \\ (i = 0)\ y_1 &= (1 + 2 \cdot .5^2 \cdot 0) y_0 + .5 \\ &= 1.5 \\ (i = 1)\ y_2 &= (1 + 2 \cdot .5^2 \cdot 1) y_1 + .5 \\ &= 1.5 \cdot 1.5 + .5 \\ &= 2.75 \end{aligned}$$