Problem 1 (30 pts) Solve the following initial value problems (’ represents $\frac{d}{dt}$).

(a) $y'' - y = 0 \quad y(0) = 1, \quad y'(0) = -1$.

Solution: The characteristic equation $r^2 - 1 = 0$ has roots $\pm 1$ so we get:

$$y = c_1 e^t + c_2 e^{-t}$$

now solving for the initial conditions gives:

$$y(0) = c_1 + c_2 = 1$$
$$y'(0) = c_1 - c_2 = -1$$

$\implies c_1 = 0, c_2 = 1$

so the solution is

$$y = e^{-t}$$

(b) $y'' + 2y' + y = 0 \quad y(0) = 1, \quad y'(0) = 0$.

Solution: The characteristic equation $r^2 + 2r + 2 = 0$ has one repeated root $r = -1$.

Thus the general solution has the form:

$$y = c_1 e^{-t} + c_2 te^{-t} = (c_1 + c_2 t)e^{-t}$$

now solving for the initial conditions gives:

$$y(0) = c_1 = 1$$
$$y'(0) = c_1 - c_2 = 0$$

$\implies c_1 = 1, c_2 = 1$

so the solution is

$$y = (1 + t)e^{-t}$$

(c) $y'' + y' + y = 0 \quad y(0) = 0, \quad y'(0) = 1$

Solution: The characteristic equation $r^2 + r + 1 = 0$ has roots $r = \frac{-1 \pm \sqrt{3}i}{2}$. Thus the solution has the general form:

$$y = e^{-\frac{t}{2}} \left( c_1 \cos \frac{\sqrt{3}}{2} t + c_2 \sin \frac{\sqrt{3}}{2} t \right)$$

Solving for the initial conditions we get

$$y(0) = c_1 = 0 \implies y = c_2 e^{-\frac{t}{2}} \sin \frac{\sqrt{3}}{2} t$$
Problem 2 (20 pts) Let $y_1$ and $y_2$ be a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$. Then, prove that $y_3 = y_1 + y_2$ and $y_4 = y_1 - y_2$ also form a fundamental set of solutions to this differential equation.
Solution: Consider the Wronskian of $y_3$ and $y_4$:

$$W(y_3, y_4)(t) = \begin{vmatrix} y_3 & y_4 \\ y'_3 & y'_4 \end{vmatrix} = \begin{vmatrix} y_1 + y_2 & y_1 - y_2 \\ y'_1 + y'_2 & y'_1 - y'_2 \end{vmatrix} = (y_1 + y_2)(y'_1 - y'_2) - (y'_1 + y'_2)(y_1 - y_2) = -2(y_1y'_2 - y'_1y_2) - 2(y_1y'_2 - y'_1y_2) = -2W(y_1, y_2)(t) \neq 0$$

since $y_1$ and $y_2$ form a fundamental set of solutions. Thus we have that $W(y_3, y_4)(t) \neq 0$ so $y_3$ and $y_4$ are a fundamental set of solutions.

Score of this page: ______________

Problem 3 (20 pts) Find the general solution to the following nonhomogeneous differential equation by the method of undetermined coefficients.

$$y'' + y' - 6y = e^{3t}.$$ 

Solution: First let’s solve the homogeneous equation:

$$y'' + y' - 6y = 0$$

The characteristic equation is $r^2 + r - 6 = 0$ which has roots $r = 2, -3$. Thus

$$y_1 = e^{2t} \text{ and } y_2 = e^{-3t}$$

Now assume $Y = Ae^{3t}$ since 3 is not a root of the characteristic equation, so:

$$Y' = 3Ae^{3t} \text{ and } Y'' = 9Ae^{3t}$$

so

$$Y'' + Y' - 6Y = Ae^{3t}(9 + 3 - 6) = e^{3t} \implies A = \frac{1}{6}$$

thus:

$$y = c_1e^{2t} + c_2e^{-3t} + \frac{1}{6}e^{3t}$$
Select one of the following problems (either Problem 4-1 or 4-2), and solve it. You don’t have to solve both of them, but if you want to and can solve both of them, you will get extra points.

**Problem 4-1** (30 pts) Consider the following initial value problem.

\[(t - 1)y'' - ty' + y = 0, \quad t < 1, \quad y(0) = 1, y'(0) = 0.\]

(a) (10 pts) Find a particular solution \(y_1(t)\) by observation/inspection. [Hint: This is a very simple function of \(t\).]

*Solution:* You can see both \(y_1 = t\) and \(y_1 = e^t\) are solutions.

(b) (10 pts) Find a second linearly independent solution \(y_2(t)\).

*Solution:* Assume that \(y_2\) has the form \(y_2(t) = v(t)y_1(t)\)

If you take \(y_1 = t\) then we have

\[y_2(t) = v(t)y_1(t) = v(t) \cdot t\]

then

\[y'_2 = v't + v\]
\[y''_2 = v''t + 2v'\]

Plugging into the ODE we get:

\[(t - 1)(v''t + 2v') - t(v't + v) + vt = 0\]

which simplifies to

\[
\frac{v''}{v'} = \frac{t^2 - 2t + 2}{t(t - 1)} = 1 - \frac{2}{t} + \frac{1}{t - 1}
\]

\[\Rightarrow \ln |v'| = t - 2 \ln |t| + \ln |t - 1|\]

\[\Rightarrow v' = \frac{t - 1}{t^2} e^t = \frac{1}{t} e^t - \frac{1}{t^2} e^t\]

Now we integrate to get \(v\):

\[v = \int \frac{1}{t} e^t dt - \int \frac{1}{t^2} e^t dt = \int \frac{1}{t} e^t dt - \left( -\frac{1}{t} e^t + \int \frac{1}{t} e^t dt \right) = \frac{1}{t} e^t\]

So we have

\[y_2 = vt = \frac{1}{t} e^t \cdot t = e^t\]
Giving the general solution:
\[ y(t) = c_1 e^t + c_2 t \]

Alternatively: we could take \( y_1 = e^t \) then
\[ y_2(t) = v(t)y_1(t) = v(t)e^t \]
so that
\[ y'_2 = (v' + v)e^t \]
\[ y''_2 = (v'' + 2v' + v)e^t \]

Substituting back into the ODE as before we end up with the equation:
\[ \frac{v''}{v'} = -t - \frac{3}{t - 1} = -1 + \frac{1}{t - 1} \]
So that
\[ v' = (t - 1)e^{-t} \]

Integrating gives:
\[ v = \int (t - 1)e^{-t} dt = -(t - 1)e^{-t} + \int e^{-t} dt = -te^{-t} \]

Thus:
\[ y_2 = vy_1 = -te^{-t} \cdot e^t = -t \]

Giving the general solution
\[ y(t) = c_3 e^t - c_4 t \]

Notice that this matches the original solution if we take \( c_1 = c_3 \) and \( c_2 = -c_4 \).

(c) (10 pts) Solve the initial value problem.
Solution:
\[ y = c_1 e^t + c_2 t \]
So
\[ y(0) = c_1 = 1 \]
\[ y'(0) = c_1 + c_2 = 0 \]
\[ \implies c_1 = 1 \text{ and } c_2 = -1 \]
So
\[ y(t) = e^t - t \]
Problem 4-2 (30 pts) A position $x(t)$ of mass $m$ attached to a very simple spring system with the spring constant $k > 0$ can be written as:

$$mx'' + kx = 0.$$ 

(a) (10 pts) Suppose the initial condition is $x(0) = 0$ and $x'(0) = 1$. Solve this equation.

Solution: The characteristic equation $mr^2 + k = 0$ has roots $r = \pm i \sqrt{\frac{k}{m}}$. Thus we have a solution of the form

$$x(t) = c_1 \cos \sqrt{\frac{k}{m}} t + c_2 \sin \sqrt{\frac{k}{m}} t.$$

Now, satisfying the initial conditions we get

$$x(0) = c_1 = 0 \quad x'(0) = \sqrt{\frac{k}{m}} c_2 = 1 \quad \Rightarrow \quad c_1 = 0 \quad \text{and} \quad c_2 = \sqrt{\frac{m}{k}}.$$ 

Thus

$$x(t) = \sqrt{\frac{m}{k}} \sin \sqrt{\frac{k}{m}} t.$$

(b) (20 pts) If the external force $\sin \omega t$ is applied, then the differential equation becomes:

$$mx'' + kx = \sin \omega t.$$ 

At what value of $\omega$ the solution becomes unbounded (i.e., blows up) as $t \to \infty$? State your reasoning clearly. (This situation is called a resonance in physics.)

Solution: We know that if $\omega i$ is a solution to the characteristic equation $mr^2 + k = 0$ then the particular solution is of the form

$$x(t) = At \cos \omega t + Bt \sin \omega t,$$

which blows up as $t \to \infty$. Thus $\omega = \pm \sqrt{\frac{k}{m}}$. 

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