MATH 22B-001: Differential Equations
MidTerm Exam II, Wednesday February 21, 2001

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Problem 1 (30 pts) Solve the following initial value problems ($'$ represents $\frac{d}{dt}$).

(a) $y'' - y = 0 \quad y(0) = 1,\ y'(0) = -1.$
Solution: The characteristic equation $r^2 - 1 = 0$ has roots ± 1 so we get:

$$y = c_1 e^t + c_2 e^{-t}$$

now solving for the initial conditions gives:

$$\left.\begin{array}{l} y(0) = c_1 + c_2 = 1 \\ y'(0) = c_1 - c_2 = -1 \end{array}\right\} \Longrightarrow c_1 = 0, c_2 = 1$$

so the solution is

$$y = e^{-t}$$

(b) $y'' + 2y' + y = 0 \quad y(0) = 1,\ y'(0) = 0.$
Solution:The characteristic equation $r^2 + 2r + 2 = 0$ has one repeated root $r = -1$. Thus the general solution has the form:

$$y = c_1 e^{-t} + c_2 t e^{-t} = (c_1 + c_2 t)e^{-t}$$

now solving for the initial conditions gives:

$$\left.\begin{array}{l} y(0) = c_1 = 1 \\ y'(0) = c_1 - c_2 = 0 \end{array}\right\} \Longrightarrow c_1 = 1, c_2 = 1$$

so the solution is

$$y = (1 + t)e^{-t}$$

(c) $y'' + y' + y = 0 \quad y(0) = 0,\ y'(0) = 1$
Solution:The characteristic equation $r^2 + r + 1 = 0$ has roots $r = \frac{-1 \pm \sqrt{3}i}{2}$. Thus the solution has the general form:

$$y = e^{-\frac{t}{2}} \left(c_1 \cos \frac{\sqrt{3}}{2} t + c_2 \sin \frac{\sqrt{3}}{2} t \right)$$

Solving for the initial conditions we get

$$y(0) = c_1 = 0 \Longrightarrow y = c_2 e^{-\frac{t}{2}} \sin \frac{\sqrt{3}}{2} t$$

$$y'(0) = c_2\frac{\sqrt{3}}{2} = 1 \Longrightarrow c_2 = \frac{2}{\sqrt{3}}$$

Thus

$$y = \frac{2}{\sqrt{3}}e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t$$

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Problem 2 (20 pts) Let y_1 and y_2 be a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$. Then, prove that $y_3 = y_1 + y_2$ and $y_4 = y_1 - y_2$ also form a fundamental set of solutions to this differential equation.

Solution:Consider the Wronskian of y_3 and y_4:

$$W(y_3, y_4)(t) = \begin{vmatrix} y_3 & y_4 \\ y_3' & y_4' \end{vmatrix} = \begin{vmatrix} y_1 + y_2 & y_1 - y_2 \\ y_1' + y_2' & y_1' - y_2' \end{vmatrix}$$

$$= (y_1 + y_2)(y_1' - y_2') - (y_1' + y_2')(y_1 - y_2) = -2(y_1y_2' - y_1'y_2)$$

$$-2(y_1y_2' - y_1'y_2) = -2W(y_1, y_2)(t) \neq 0$$

since y_1 and y_2 form a fundamental set of solutions. Thus we have that $W(y_3, y_4)(t) \neq 0$ so y_3 and y_4 are a fundamental set of solutions.

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Problem 3 (20 pts) Find the general solution to the following nonhomogeneous differential equation by the method of undetermined coefficients.

$$y'' + y' - 6y = e^{3t}.$$

Solution: First let's solve the homogeneous equation:

$$y'' + y' - 6y = 0$$

The characteristic equation is $r^2 + r - 6 = 0$ which has roots $r = 2, -3$. Thus

$$y_1 = e^{2t} \;\; and \;\; y_2 = e^{-3t}$$

Now assume $Y = Ae^{3t}$ since 3 is not a root of the characteristic equation, so:

$$Y' = 3Ae^{3t} \;\; and \;\; Y'' = 9Ae^{3t}$$

so

$$Y'' + Y' - 6Y = Ae^{3t}(9 + 3 - 6) = e^{3t} \Longrightarrow A = \frac{1}{6}$$

thus;

$$y = c_1e^{2t} + c_2e^{-3t} + \frac{1}{6}e^{3t}$$

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Select one of the following problems (either Problem 4-1 or 4-2), and solve it. You don't have to solve both of them, but if you want to and can solve both of them, you will get extra points.

Problem 4-1 (30 pts) Consider the following initial value problem.

$$(t-1)y'' - ty' + y = 0, \quad t < 1, \quad y(0) = 1, y'(0) = 0.$$

(a) (10 pts) Find a particular solution $y_1(t)$ by observation/inspection. [Hint: This is a very simple function of t.]
Solution:You can see both $y_1 = t$ and $y_1 = e^t$ are solutions.

(b) (10 pts) Find a second linearly independent solution $y_2(t)$.
Solution:Assume that y_2 has the form

$$y_2(t) = v(t)y_1(t)$$

If you take $y_1 = t$ then we have

$$y_2(t) = v(t)y_1(t) = v(t) \cdot t$$

then

$$\begin{aligned} y_2' &= v't + v \\ y_2'' &= v''t + 2v' \end{aligned}$$

Plugging into the ODE we get:

$$(t-1)(v''t + 2v') - t(v't + v) + vt = 0$$

which simplifies to

$$\begin{aligned} \frac{v''}{v'} &= \frac{t^2 - 2t + 2}{t(t-1)} = 1 - \frac{2}{t} + \frac{1}{t-1} \\ \Rightarrow\ \ln|v'| &= t - 2\ln|t| + \ln|t-1| \\ \Rightarrow\ v' &= \frac{t-1}{t^2}e^t = \frac{1}{t}e^t - \frac{1}{t^2}e^t \end{aligned}$$

Now we integrate to get v:

$$v = \int \frac{1}{t}e^t dt - \int \frac{1}{t^2}e^t dt = \int \frac{1}{t}e^t dt - \left(-\frac{1}{t}e^t + \int \frac{1}{t}e^t dt\right) = \frac{1}{t}e^t$$

So we have

$$y_2 = vt = \frac{1}{t}e^t \cdot t = e^t$$

Giving the general solution:

$$y(t) = c_1 e^t + c_2 t$$

Alternatively: we could take $y_1 = e^t$ then

$$y_2(t) = v(t)y_1(t) = v(t)e^t$$

so that

$$y_2' = (v' + v)e^t$$
$$y_2'' = (v'' + 2v' + v)e^t$$

Substituing back into the ODE as before we end up witht the equation:

$$\frac{v''}{v'} = -\frac{t-2}{t-1} = -1 + \frac{1}{t-1}$$

So that

$$v' = (t-1)e^{-t}$$

Integrating gives:

$$v = \int (t-1)e^{-t}dt = -(t-1)e^{-t} + \int e^{-t}dt = -te^{-t}$$

Thus:

$$y_2 = vy_1 = -te^{-t} \cdot e^t = -t$$

Giving the general solution

$$y(t) = c_3 e^t - c_4 t$$

Notice that this matches the original solution if we take $c_1 = c_3$ and $c_2 = -c_4$.

(c) (10 pts) Solve the initial value problem.
Solution:

$$y = c_1 e^t + c_2 t$$

So

$$\left.\begin{array}{l} y(0) = c_1 = 1 \\ y'(0) = c_1 + c_2 = 0 \end{array}\right\} \Longrightarrow c_1 = 1 \;\; and \;\; c_2 = -1$$

So

$$y(t) = e^t - t$$

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Problem 4-2 (30 pts) A position $x(t)$ of mass m attached to a very simple spring system with the spring constant $k > 0$ can be written as:

$$mx'' + kx = 0.$$

(a) (10 pts) Suppose the initial condition is $x(0) = 0$ and $x'(0) = 1$. Solve this equation.

Solution:The characteristic equation $mr^2 + k = 0$ has roots $r = \pm i\sqrt{\frac{k}{m}}$. Thus we have a solution of the form

$$x(t) = c_1 \cos\sqrt{\frac{k}{m}}t + c_2 \sin\sqrt{\frac{k}{m}}t$$

Now, satisfying the initial conditions we get

$$\left.\begin{array}{l} x(0) = c_1 = 0 \\ x'(0) = \sqrt{\frac{k}{m}}c_2 = 1 \end{array}\right\} \Longrightarrow c_1 = 0 \ \ and \ \ c_2 = \sqrt{\frac{m}{k}}$$

Thus

$$x(t) = \sqrt{\frac{m}{k}} \sin\sqrt{\frac{k}{m}}t$$

(b) (20 pts) If the external force $\sin\omega t$ is applied, then the differential equation becomes:

$$mx'' + kx = \sin\omega t.$$

At what value of ω the solution becomes unbounded (i.e., blows up) as $t \to \infty$? State your reasoning clearly. (This situation is called a *resonance* in physics.)

Solution:We know that if ωi is a solution to the characteristic equation $mr^2 + k = 0$ then the particular solution os of the form

$$x(t) = At\cos\omega t + Bt\sin\omega t$$

which blows up as $t \longrightarrow \infty$.Thus $\omega = \pm\sqrt{\frac{k}{m}}$.