Drums That Sound the Same

S. J. Chapman

1. INTRODUCTION. In 1966 Kac [7] asked the question "Can you hear the shape of a drum?", that is, if you know the frequencies at which a drum vibrates, can you determine it's shape? Mathematically this corresponds to the following problem. If u is the displacement of a membrane D from its mean position, then u satisfies

$$\nabla^2 u = \frac{\partial^2 u}{\partial t^2}, \text{ in } D, \tag{1}$$

$$u = 0, \text{ on } \partial D. \tag{2}$$

Seeking a solution by separation of variables $u(x, y, t) = \psi(t)\phi(x, y)$ yields

$$\frac{\phi_{xx} + \phi_{yy}}{\phi} = \frac{\psi_{tt}}{\psi} = \text{constant} = \lambda, \text{ say.}$$

Hence

$$u = \sin(\sqrt{\lambda}\, t)\phi(x, y), \tag{3}$$

where

$$\nabla^2 \phi + \lambda \phi = 0, \text{ in } D, \tag{4}$$

$$\phi = 0, \text{ on } \partial D. \tag{5}$$

This is an eigenvalue problem: there exists a nonzero solution ϕ only for certain values of λ known as eigenvalues. The set of eigenvalues is known as the

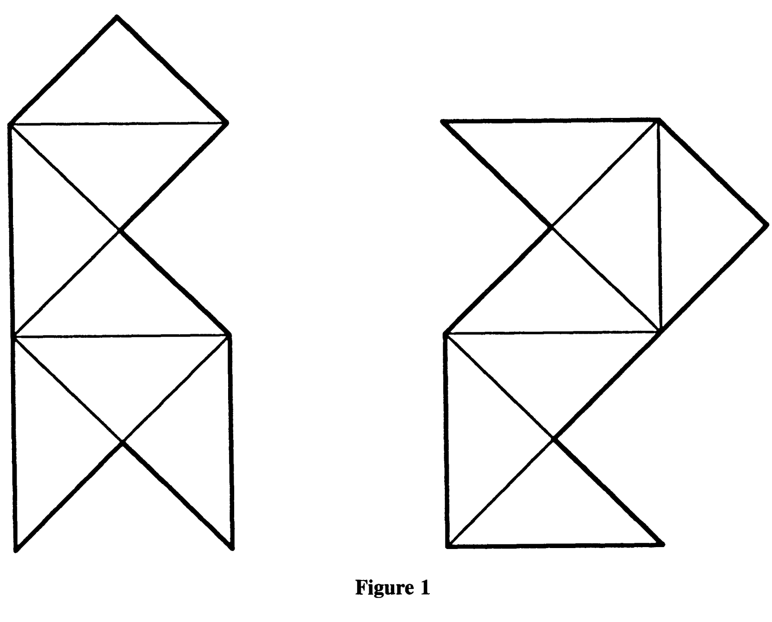
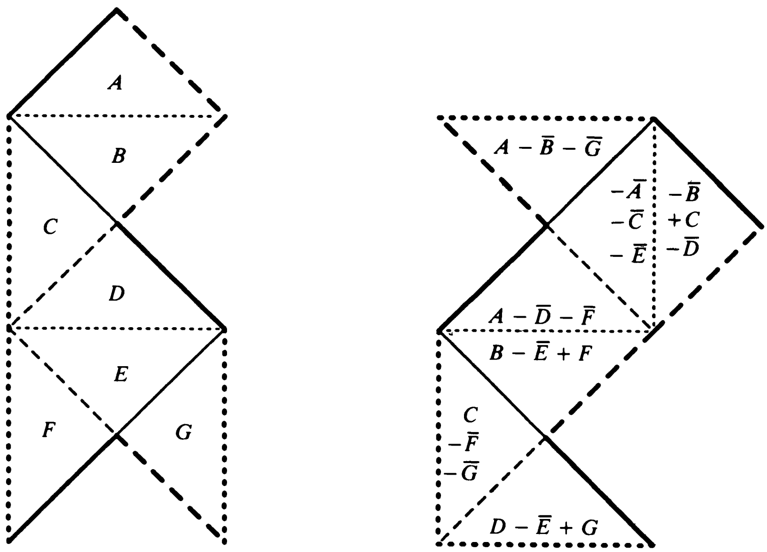


Figure 1

eigenvalue spectrum, and is discrete in this case. We see by equation (3) that the eigenvalues λ are the squares of the frequencies of vibration, and that each eigensolution can be viewed as a standing wave on the domain D. The general solution of (1), (2) is a superposition of these special solutions.

Kac's question is now the following: are two domains with the same eigenvalue spectrum (where the eigenvalues are counted with multiplicities) necessarily congruent?

It has been shown that the eigenvalues do determine certain properties of D, for example the area, the circumference, and the number of connected components [**7**]. However, the answer to the question is in fact no. Figure 1 shows an example of two domains with exactly the same eigenvalue spectrum which was given by Gordon et al. [**5**] (see also [**6**]).



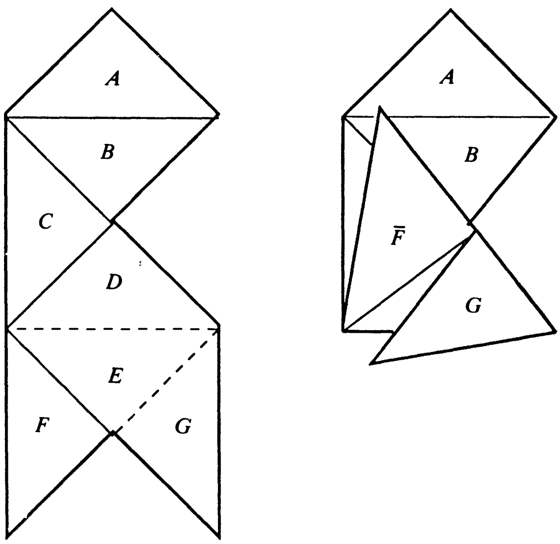
(a) (b)

Figure 2

A simple proof that the eigenvalues are identical has been given by Berard [**3**] (see also [**1**, **2**]), who constructs the map shown in Figure 2, which takes an eigenfunction for the first domain and maps it onto an eigenfunction for the second domain, with the same eigenvalue λ. Here $A + B$ means that to obtain the value of the function in that triangle we add the values of the function at the corresponding points in triangles A and B. We have used different types of lines for the edges of the triangles to help make it clear how each should be orientated when making this identification. In some cases it is necessary to reflect the triangle about its line of symmetry, and this we have indicated by $\bar{A}$. Only the zero function maps to the zero function, which implies that for any eigenfunction of the first eigenvalue problem there is a corresponding eigenfunction of the second eigenvalue problem, with the same eigenvalue λ. Thus any eigenvalue of the first problem is also an eigenvalue of the second problem (including multiplicities). A similar map of an eigenfunction of the second problem to one of the first shows that any eigenvalue of the second problem is also an eigenvalue of the first, and therefore the two domains are isospectral.

Here we give an interpretation of the transposition of a solution of the first problem to one of the second problem in terms of paper folding. This will allow us to generate many new isospectral domains, including a simple example in which the eigenvalues can be calculated explicitly. We note that the method of transposition has also recently been used by Buser et al. [4] to generate new examples of isospectral plane domains.

2. PAPER FOLDING. Consider a paper cutout of a domain, and a function which is zero on the boundary of the domain. We now fold the paper to create a new domain. We define a function, the transposition, on the new domain, by adding the values of the original function at points that lie on top of one another, with the convention that if the paper is reversed then the function is subtracted rather than added. This function will automatically be zero on the boundary of the new domain, since it will be zero along any fold of the paper, as well as on any edge. The reader may find it helpful in what follows to actually construct the shapes by folding paper.



(a) (b)

Figure 3

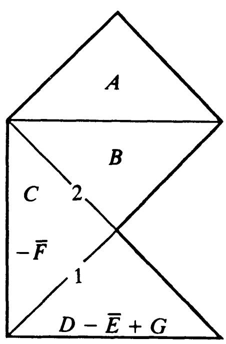


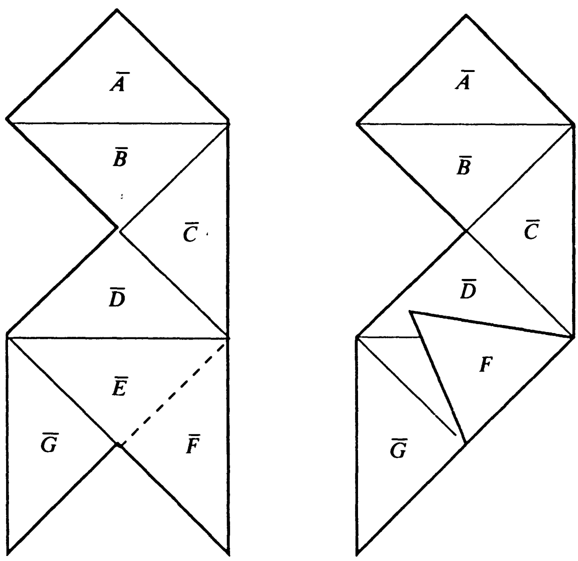
Figure 4

For example consider the domain shown in Figure 2a, and a function on this domain which is zero on the boundary. We label the triangles A to G on the front and $\bar{A}$ to $\bar{G}$ on the back and fold the domain as shown in Figure 3 (where a dotted line indicates a fold of the paper). We then obtain a function on the domain shown in Figure 4 which is zero on the boundary.

We now take several copies of the original domain D and fold them to create domains D_1, D_2, etc. We glue these together to create a new domain D^*, and define the transposition on this domain to be the sum of the transpositions on D_1, D_2, etc. Now, if we can glue the domains D_1, D_2, etc. together in such a way that the first derivative of the transposition is continuous, then we will have actually created an eigenfunction on the new domain D^*.[1]

In order for the first derivative of the transposition to be continuous it is sufficient that

(1) Every fold lies along an outside edge of the new shape.
(2) Each edge of each copy of the original shape that lies in the interior of the final shape must be adjacent to its reflection on an associated copy of the original shape.



(a) (b)

Figure 5

For example, the first derivative of the transposition is discontinuous across the lines (1) and (2) in Figure 4. However, if we add the same initial drum shape folded as shown in Figure 5 then we ensure continuity of the first derivative across the lines (1) and (2) (Figure 6), though the first derivative is now discontinuous

[1]The resulting function is once differentiable and satisfies the eigenequation except possibly on the seams. Such a function is a weak solution of the equation and therefore by elliptic regularity a strong solution; see G. Folland, Introduction to PDE, PUP, 1976: specifically apply Corollary 6.28 repeatedly and then Corollary 6.10 to find that the transposed function is indeed an eigenfunction.

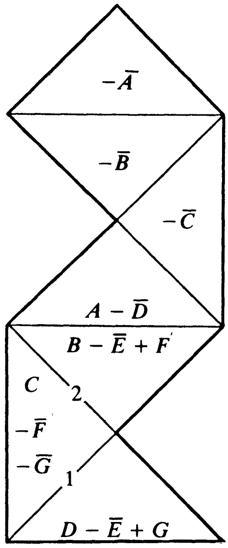


Figure 6

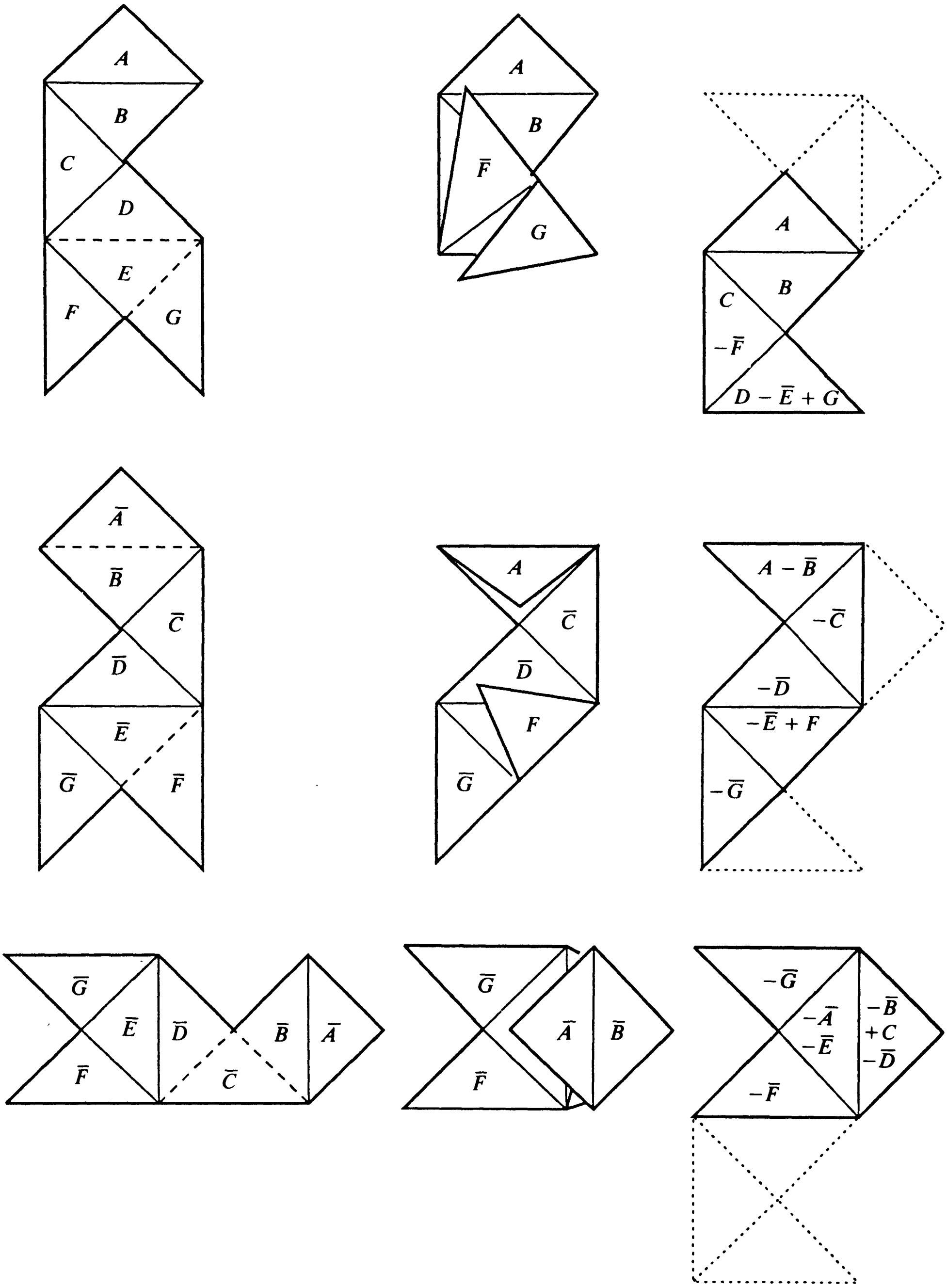
across other lines. Note that the continuity of the transposed function itself is automatic since the original function is zero on the boundary of each component.

Figure 7 shows how the pieces fit together for the example given in the introduction. Three copies of the original shape (Fig. 2a) are folded along the dotted lines shown in Fig. 7a, to give the shapes shown in Fig. 7c (Figure 7b shows a three dimensional view of how each piece will look before it is squashed flat). These shapes are then superimposed to create the shape shown in Fig. 2b (the dotted lines in Fig. 7c indicate the position of each component in the new shape).

Another example of isospectral domains, and the transposition of a solution on one domain onto a solution on the other domain, is given in Figures 8 and 9. The cuts in these figures are to be interpreted as having zero width, and are shown for clarity.

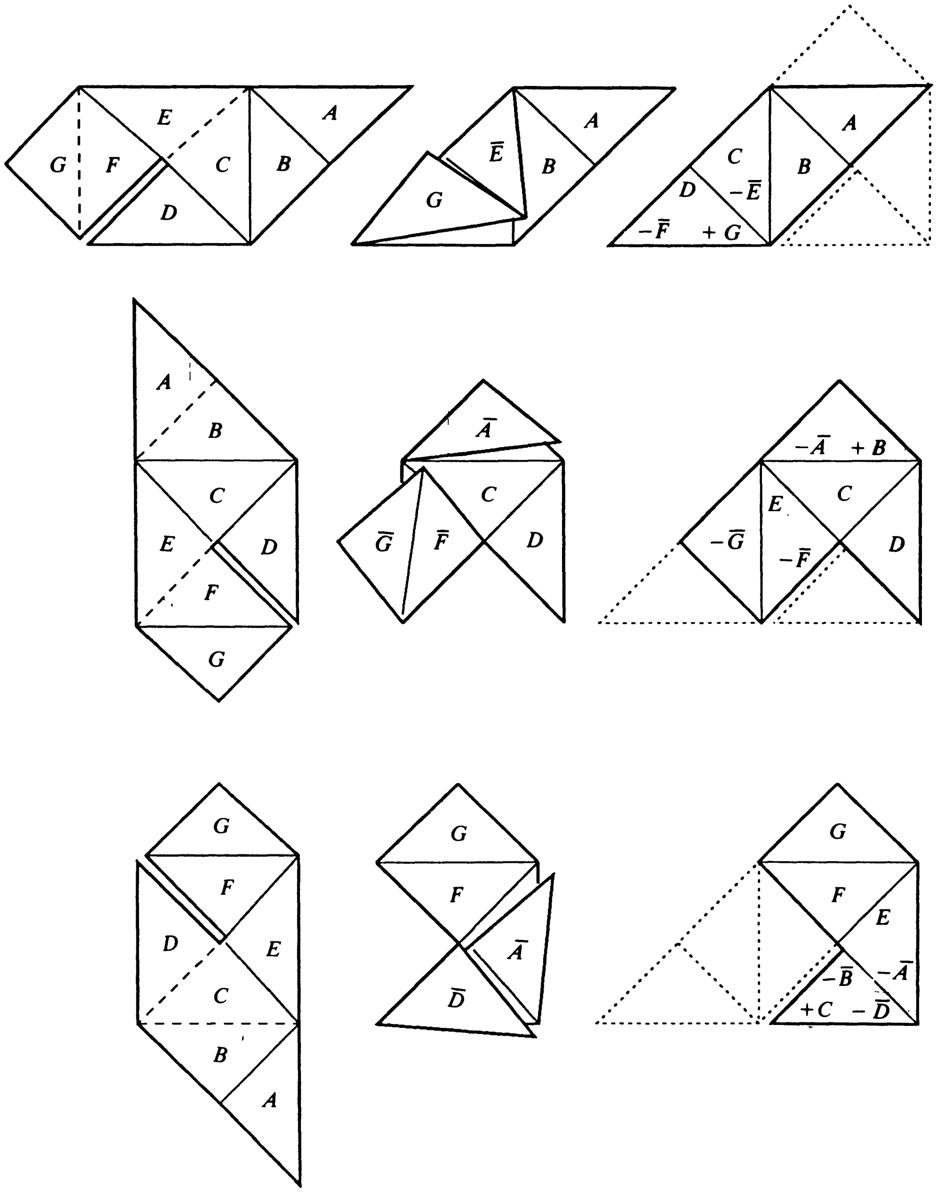
Since the method of construction of the transposition depends only on folding along the edges of the triangles, there is no need for the triangles to be right-angled. All that is important is that the two triangles adjacent to a fold lie on top of one another when the paper is folded. If we think of the shapes in Figure 2 as being constructed from a single triangle A by a series of reflections about its edges, then it is not the shape of A, but the series of reflections which is important. Choose any other triangle in place of A in Figure 10a, and perform the same series of reflections to obtain a new shape. Now place the same triangle in position d of Figure 10b, with the same orientation, and perform the series of reflections that created 10b from the basic right-angled triangle d. The two shapes obtained will then be isospectral, since the same map transposing eigenfunctions of one domain onto the other domain will work as before. For example, if we use the triangle shown in Figure 11a as our basic building block, we find that the domains shown in Figure 11b are isospectral.

It is not necessary for the triangle to be isoceles. If we take the triangle shown in Figure 12a as our basic building block, we find that the domains shown in Figure 12b are isospectral.


A
B
C
D
E
F
G
A
B
$\bar{F}$
G
A
B
C
$-\bar{F}$
$D - \bar{E} + G$
$\bar{A}$
$\bar{B}$
$\bar{C}$
$\bar{D}$
$\bar{E}$
$\bar{G}$
$\bar{F}$
A
$\bar{C}$
$\bar{D}$
F
$\bar{G}$
$A - \bar{B}$
$-\bar{C}$
$-\bar{D}$
$-\bar{E} + F$
$-\bar{G}$
$\bar{G}$
$\bar{E}$
$\bar{D}$
$\bar{B}$
$\bar{A}$
$\bar{F}$
$\bar{C}$
$\bar{G}$
$\bar{A}$
$\bar{B}$
$\bar{F}$
$-\bar{G}$
$-\bar{A}$
$-\bar{E}$
$-\bar{B}$
$+C$
$-\bar{D}$
$-\bar{F}$

(a) (b) (c)

Figure 7


E
A
G
F
C
B
D
$\bar{E}$
A
B
G
A
C
B
D
$-\bar{E}$
$-\bar{F}$
$+G$
A
B
C
E
D
F
G
$\bar{A}$
C
$\bar{G}$
$\bar{F}$
D
$-\bar{A}$
$+B$
C
E
$-\bar{G}$
$-\bar{F}$
D
G
F
D
E
C
B
A
G
F
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G
F
E
$-\bar{B}$
$-\bar{A}$
$+C$
$-\bar{D}$

Figure 8

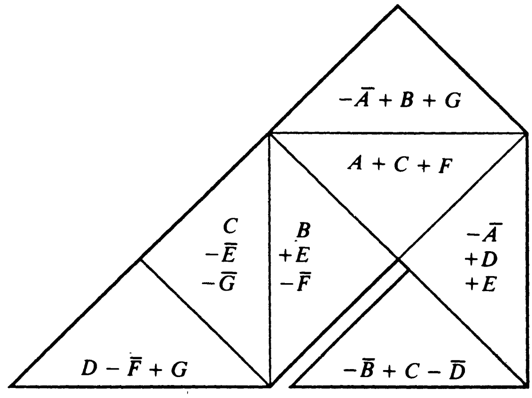


Figure 9

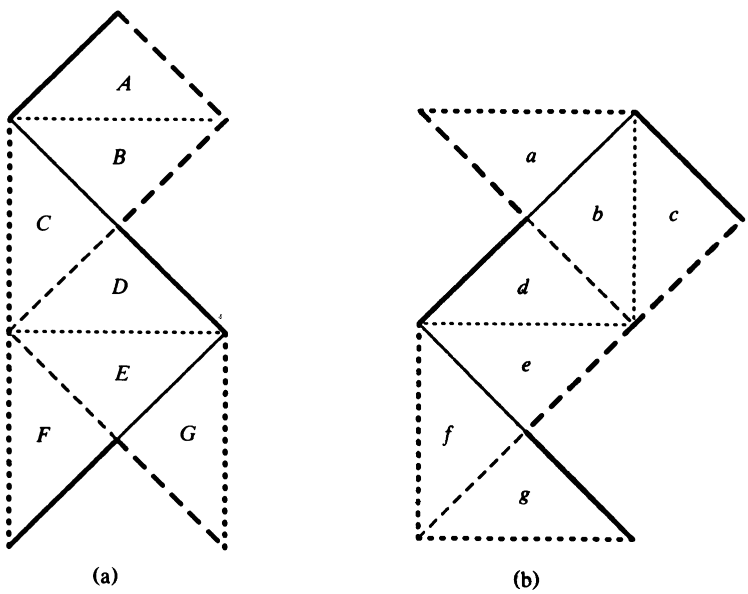


Figure 10

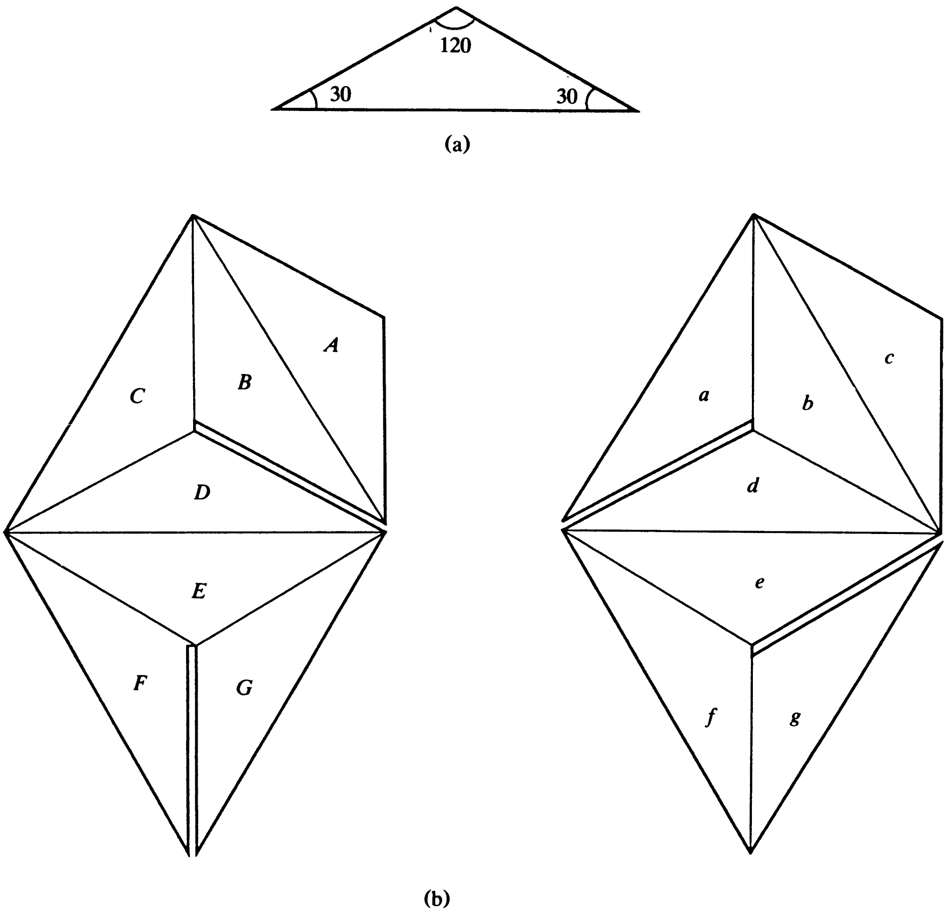


Figure 11

By the same considerations, it is not even necessary for the basic shape to be a triangle. Any shape with at least three edges will do. We simply choose three edges to represent the three sides of the triangle, about which we will reflect the shape. If we then follow the same pattern of reflection that created the original shapes of Fig. 2 from the basic right-angled triangle, then we have again isospectral drums. The example shown in Figure 13 uses squares.

In fact, the basic starting shape can be as complicated as you like. To construct different shapes, take any of the previous shapes constructed of triangles, squares etc. and fold both shapes until a single triangle (for example) remains. Place the two resulting triangles on top of one another (with the correct orientation, i.e. so that the solid, dotted and dashed lines match up). Now cut out shapes as when making paper dolls. The shapes obtained when the paper is unfolded will again have exactly the same eigenvalues as each other, since the same one-to-one correspondence between solutions will hold as before. We note that the cutout will be in one piece if and only if there is a segment left uncut on each edge.

Figure 14 shows a simple example. These isospectral domains were also discovered by Gordon et al. More exotic shapes can also be made, as shown in Figure 15.

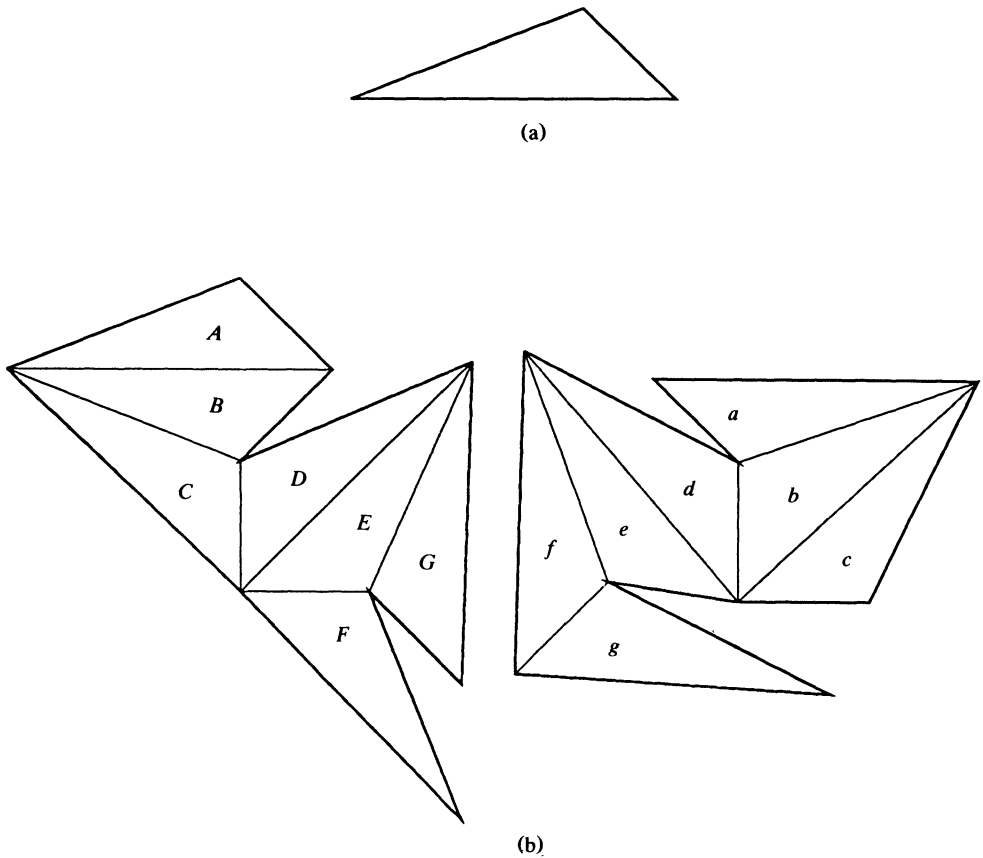


Figure 12

In this way a simpler example of drums with the same eigenvalues can be constructed—one in which the eigenvalues can actually be calculated explicitly. Consider the original example, folded and cut along one edge as shown in Figure 16a. Then the drums obtained are as shown in Figure 16b. Discarding the single small triangle, which appears once in each drum, we have the domains shown in Figure 17. The spectrum of each of the disconnected domains in Figure 17 is equal to the union of the spectra of each of the components (with the multiplicity of an eigenvalue being equal to the sum of its multiplicities in the components), since each of the components vibrates independently. The eigenfunctions for a rectangle of length a and width b are

$$\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}, \qquad n, m \text{ integers,}$$

with corresponding eigenvalues $\lambda = \pi^2((n/a)^2 + (m/b)^2)$. For a right-angled isoceles triangle with short sides of length c the eigenfunctions are

$$\sin\frac{i\pi x}{c}\sin\frac{j\pi y}{c} - \sin\frac{j\pi x}{c}\sin\frac{i\pi y}{c}, \qquad i, j \text{ integers}, \quad i > j,$$

with corresponding eigenvalues $\lambda = \pi^2((i/c)^2 + (j/c)^2)$. Thus we find that the eigenvalues for each domain are as shown in Fig. 17. We shall now show that the

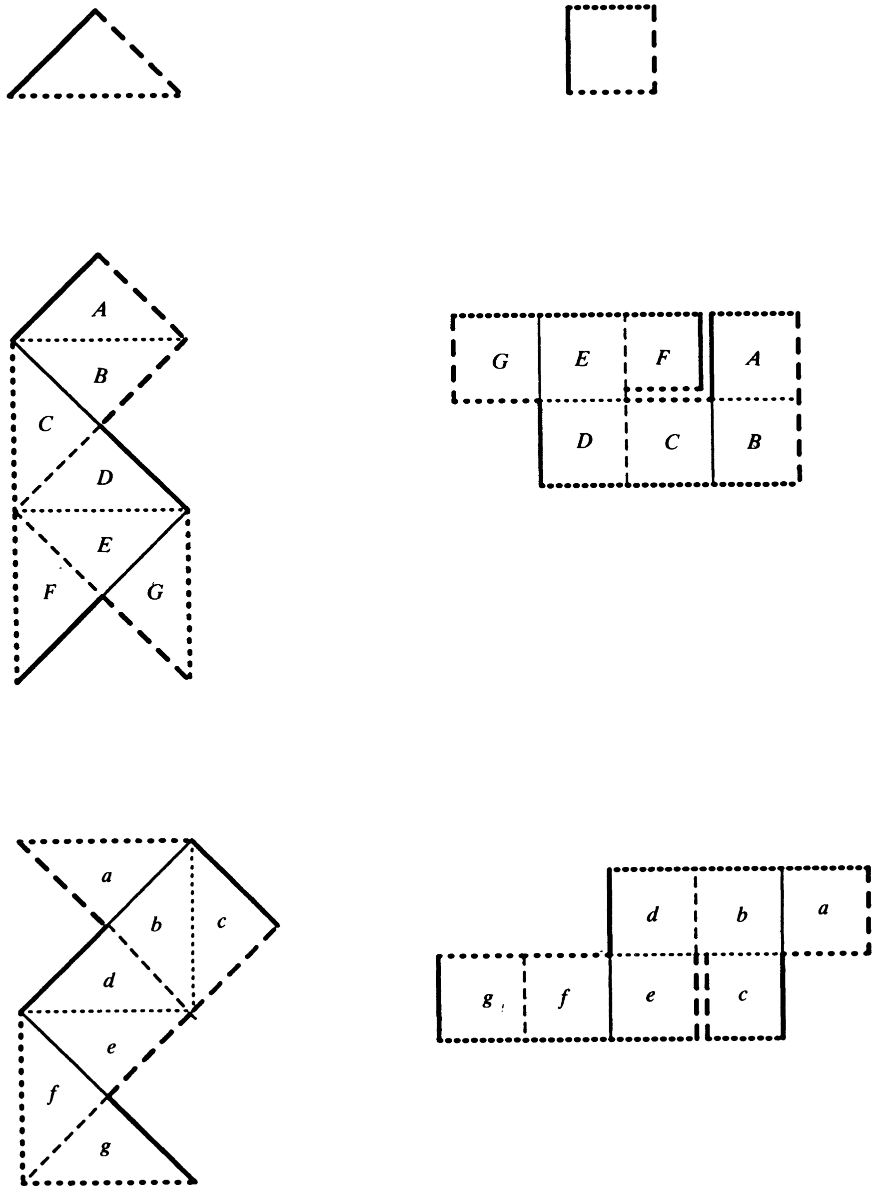


Figure 13

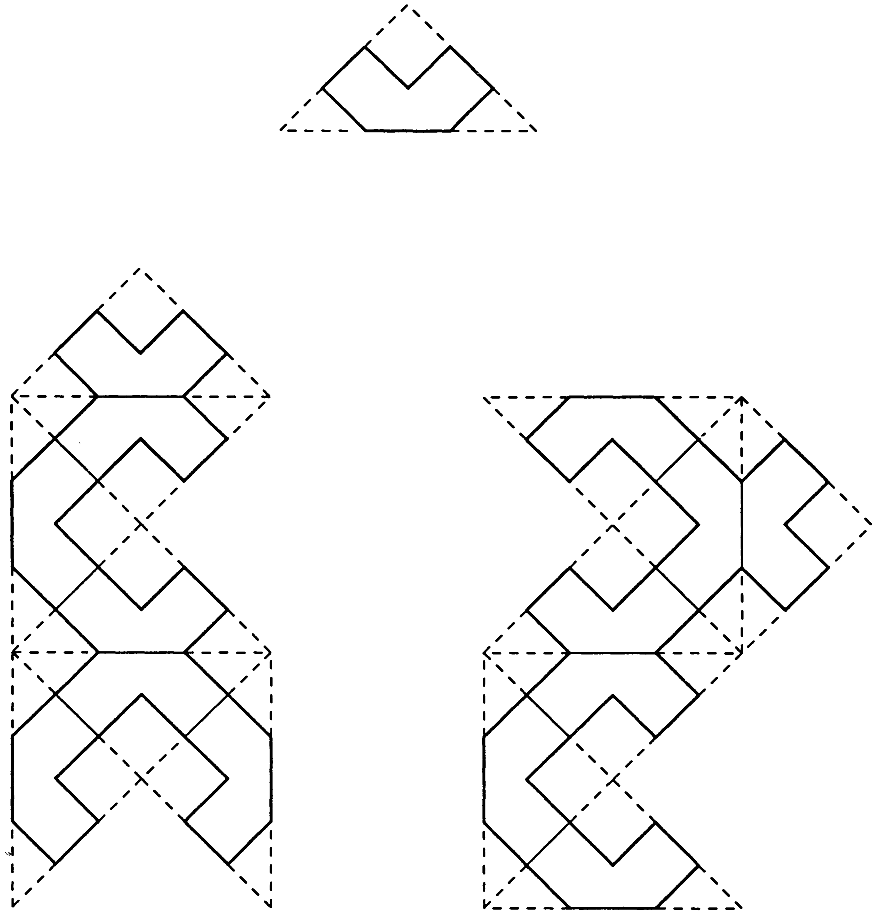


Figure 14

combined eigenvalues of the two domains of Figure 17a are identical to the combined eigenvalues of the two domains of Figure 17b.

With N even we set $N = 2n$ and $M = m$. Then we have $(N/2)^2 + M^2 = n^2 + m^2$. When N is odd we set $i = \max(N, 2M)$, $j = \min(N, 2M)$. Then $(N/2)^2 + M^2 = (i/2)^2 + (j/2)^2$, and $i > j$. This takes care of the eigenvalues in which one of i, j is even and the other is odd. If we set $i = I + J$ and $j = I - J$ then $(I^2 + J^2)/2 = (i/2)^2 + (j/2)^2$, and $i > j$. This takes care of the eigenvalues in which either i and j are both even or both odd.

Finally, we note that the same procedure works if the boundary condition

$$u = 0 \quad \text{on } \partial D,$$

is modified to

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial D,$$

if we modify our convention and add reflected triangles instead of subtracting them. Thus all the isospectral domains found previously are also isospectral with this new boundary condition.

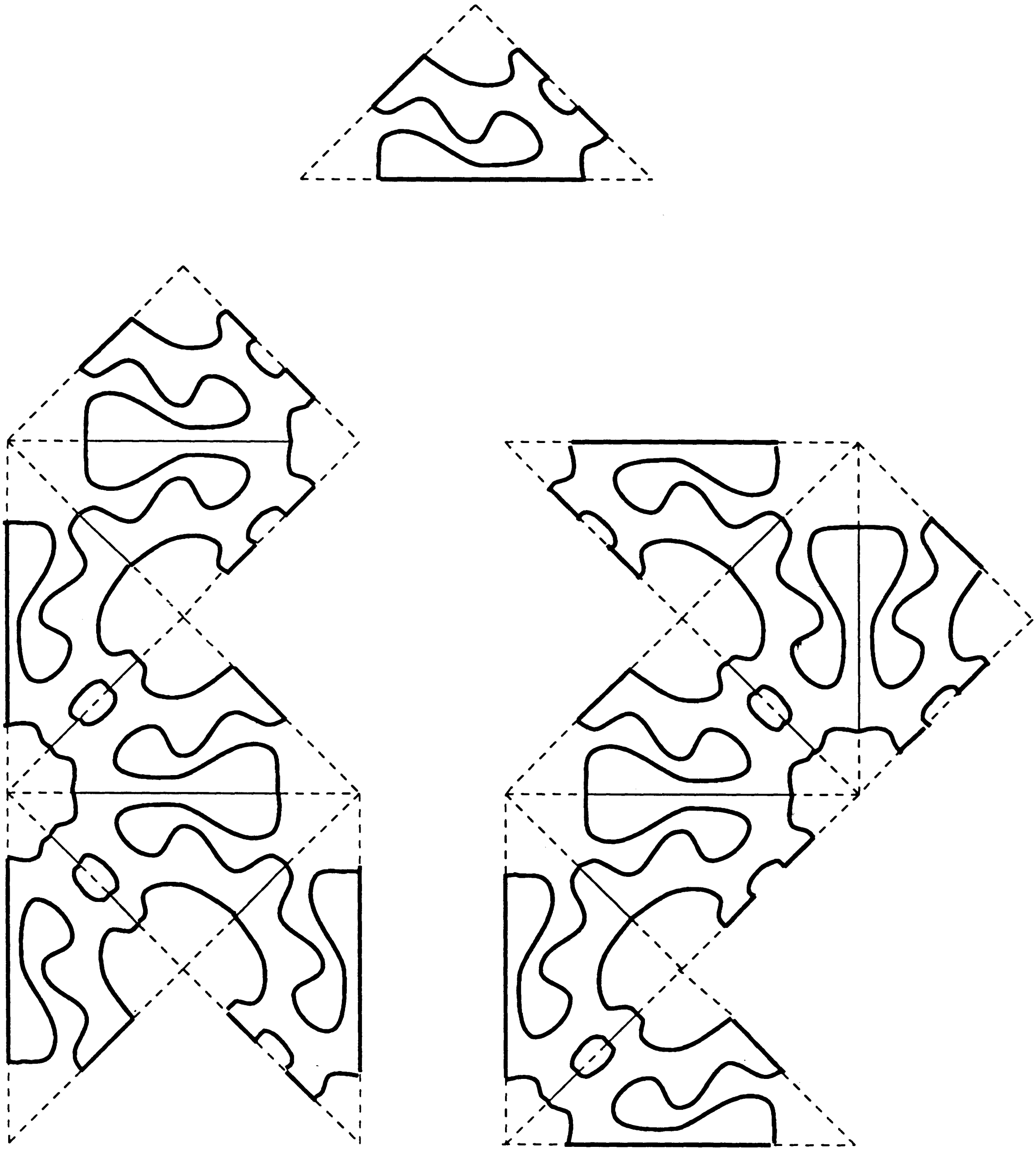
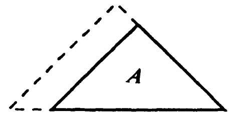
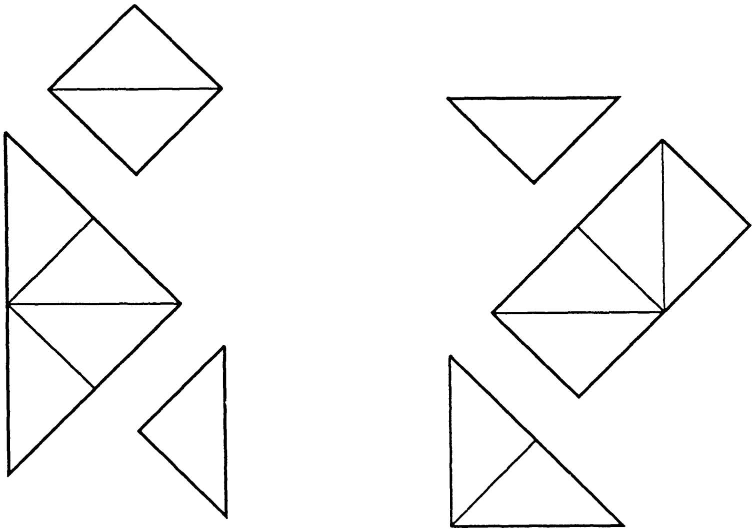


Figure 15

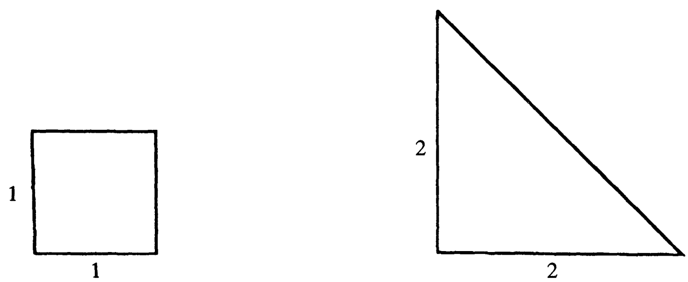


(a)



(b)

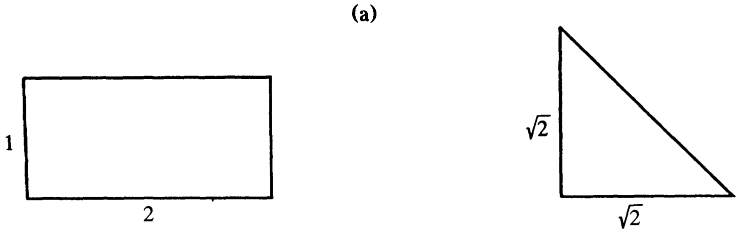
Figure 16



$\lambda = (n^2 + m^2)\pi^2$

$\lambda = \left(\left(\frac{i}{2}\right)^2 + \left(\frac{j}{2}\right)^2\right)\pi^2, \qquad i > j$

(a)



$\lambda = \left(\left(\frac{N}{2}\right)^2 + M^2\right)\pi^2$

$\lambda = \frac{1}{2}(I^2 + J^2)\pi^2, \qquad I > J$

(b)

Figure 17

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REFERENCES

1. Berard, P., Transplantation et Isospectralite I, *Math. Ann.*, 292, 547–559 (1992).
2. Berard, P., Transplantation et Isospectralite II, *Math. Ann.*, to appear.
3. Berard, P., Domaines plans isospectraux a la Gordon-Webb-Wolpert: une preure terra a terra, Preprint.
4. Buser, P., Conway, J., Doyle, P. and Semmler, K-D., Some planar isospectral domains, Preprint.
5. Gordon, C., Webb, D. and Wolpert, S., Isospectral plane domains and surfaces via Riemannian orbifolds, *Invent. Math.*, 110, 1–22 (1992).
6. Gordon, C., Webb, D. and Wolpert, S., You cannot hear the shape of a drum, *B. Am. Math. S.*, 27(1), 134–138 (1992).
7. Kac, M., Can one hear the shape of a drum?, *Am. Math. Monthly*, 73, 4, 1–23 (1966).

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PICTURE PUZZLE
(from the collection of Paul Halmos)

Half of a man and wife team
(see page 154.)