**MAT 280: Applied & Computational Harmonic Analysis Comments on Homework 1**

**Problem 1:** Everyone got this problem right!

**Problem 2:** (c) You should use the integration by parts for this problem.

**Problem 3:** There are several ways to derive the Fourier transform of the Gaussian. I believe the best way is the following.

Consider the derivative:

\[ g'(x; \sigma) = -\frac{1}{\sqrt{2\pi\sigma^3}}xe^{-x^2/2\sigma^2}. \]

\[ \Rightarrow \sigma^2 g' = -xg \]

\[ \Rightarrow \sigma^2(2\pi i\xi)\hat{g} = -\frac{i}{2\pi} \frac{d\hat{g}}{d\xi} \]

\[ \Rightarrow \frac{d\hat{g}}{d\xi} = -4\pi^2\sigma^2\xi\hat{g} \]

This is a simple ODE and we can get the solution:

\[ \hat{g}(\xi; \sigma) = Ce^{-2\pi^2\sigma^2\xi^2}. \]

But \( \hat{g}(0) = 1 \) because this is the integral of the probability density function of the normal distribution with mean 0 and variance \( \sigma^2 \). Therefore, \( C = 1. \)

\[ \hat{g}(\xi; \sigma) = e^{-2\pi^2\sigma^2\xi^2}. \]

**Problem 4:** Some people only showed the “if” part without showing the “only if” part. In fact, if you state the equality condition of the Cauchy-Schwarz inequality used in this uncertainty inequality, then it is automatically, “if and only if”. The bottom line is the Cauchy-Schwarz inequality in this case becomes:

\[ \|f\|^4 = 4 \left( \text{Re} \int xf(x)f'(x) \, dx \right)^2 \leq \int x^2|f(x)|^2 \, dx \int |f'(x)|^2 \, dx, \]

and the equality holds if and only if

\[ f'(x) = cxf(x), \quad \text{for some constant } c. \]

So, we can easily get the solution:

\[ f(x) = ae^{cx^2/2}, \quad \text{for some constants } a, c. \]

However, the function \( f \) must be in \( L^2(\mathbb{R}) \). So, we must have \( c < 0 \). Otherwise, this function cannot have a finite norm in \( L^2(\mathbb{R}) \). So, we can set \( c = -1/\sigma^2 \) for some \( \sigma > 0 \), and get the form:

\[ f(x) = ae^{-x^2/2\sigma^2}, \quad \text{for some constants } a \text{ and } \sigma > 0. \]