MAT 280: Applied & Computational Harmonic Analysis Comments on Homework 1

Problem 1: Everyone got this problem right!

Problem 2: (c) You should use the integration by parts for this problem.

Problem 3: There are several ways to derive the Fourier transform of the Gaussian. I believe the best way is the following.

Consider the derivative:

$$\begin{aligned} g'(x;\sigma) &= -\frac{1}{\sqrt{2\pi}\sigma^3} x \mathrm{e}^{-x^2/2\sigma^2}. \\ \Longrightarrow \sigma^2 g' &= -xg \\ \Longrightarrow \sigma^2(2\pi \mathrm{i}\xi)\hat{g} &= -\frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}\hat{g}}{\mathrm{d}\xi} \\ \Longrightarrow \frac{\mathrm{d}\hat{g}}{\mathrm{d}\xi} &= -4\pi^2\sigma^2\xi\hat{g} \end{aligned}$$

This is a simple ODE and we can get the solution:

$$\hat{g}(\xi;\sigma) = C\mathrm{e}^{-2\pi^2\sigma^2\xi^2}.$$

But $\hat{g}(0) = 1$ because this is the integral of the probability density function of the normal distribution with mean 0 and variance σ^2. Therefore, $C = 1$.

$$\hat{g}(\xi;\sigma) = \mathrm{e}^{-2\pi^2\sigma^2\xi^2}.$$

Problem 4: Some people only showed the "if" part without showing the "only if" part. In fact, if you state the equality condition of the Cauchy-Schwarz inequality used in this uncertainty inequality, then it is automatically, "if and only if". The bottom line is the Cauchy-Schwarz inequality in this case becomes:

$$\|f\|^4 = 4\left(\mathrm{Re}\int x\overline{f(x)}f'(x)\,\mathrm{d}x\right)^2 \le \int x^2|f(x)|^2\,\mathrm{d}x \int |f'(x)|^2\,\mathrm{d}x,$$

and the equality holds *if and only if*

$$f'(x) = cxf(x), \quad \text{for some constant } c.$$

So, we can easily get the solution:

$$f(x) = a\mathrm{e}^{cx^2/2}, \quad \text{for some constants } a, c.$$

However, the function f must be in $L^2(\mathbb{R})$. So, we must have $c < 0$. Otherwise, this function cannot have a finite norm in $L^2(\mathbb{R})$. So, we can set $c = -1/\sigma^2$ for some $\sigma > 0$, and get the form:

$$f(x) = a\mathrm{e}^{-x^2/2\sigma^2}, \quad \text{for some constants } a \text{ and } \sigma > 0.$$