MAT 280: Applied & Computational Harmonic Analysis Comments on Homework 2

Problem 1: First I have to apologize. The correct statement should be: Prove $\mathcal{F}\{\text{III}_A\}(\xi) = \frac{1}{A}\text{III}_{1/A}(\xi)$. In other words, we need the factor $1/A$ in the RHS. I also corrected my Supplementary Note II (Generalized Functions) accordingly. So, please download this corrected version from our course reference webpage.

Going back to this problem, it is important to notice that this shah (or comb) function is not really a function, but a *generalized function* or also known as a *distribution*. So, you cannot use the usual Fourier transform definition.

The best way to solve this problem is to split it into two subproblems:

(1) Show $\mathcal{F}\{\text{III}_1\}(\xi) = \text{III}_1(\xi)$;

(2) Apply the dilation operator$\boldsymbol{\delta}_A$ to $\text{III}_1(x)$ and use the the Fourier transform formula: $\mathcal{F}\{\boldsymbol{\delta}_A f\}(\xi) = \boldsymbol{\delta}_{1/A}\mathcal{F}\{f\}(\xi)$.

As for (1), I stated this in my lecture when we discussed the sampling theorem, but this is what I really wanted you to show. The correct proof goes like this. Let us write $\text{III}_1(x)$ as $\text{III}(x)$. First since the shah function is a generalized function, its Fourier transform is defined by pairing it with a very nice function ϕ in the Schwartz space § (a space of functions of infinitely many times differentiable and decaying faster than any polynomials in both space and frequency domain) as:

$$\begin{aligned}
\langle \mathcal{F}\{\text{III}\}, \phi\rangle &= \langle \text{III}, \mathcal{F}\{\phi\}\rangle \\
&= \int \text{III}(\xi)\widehat{\phi}(\xi)\,\mathrm{d}\xi \\
&= \sum_{k\in\mathbb{Z}} \widehat{\phi}(k) \\
&\overset{(a)}{=} \sum_{k\in\mathbb{Z}} \phi(k) \\
&= \int \text{III}(\xi)\phi(\xi)\,\mathrm{d}\xi.
\end{aligned}$$

Therefore, $\widehat{\text{III}}(\xi) = \text{III}(\xi)$. The equality (a) above is called the *Poisson summation formula*, and will be proved simply as follows: Consider a periodized version of $\phi(x)$, i.e.,

$$\sum_{k\in\mathbb{Z}} \phi(x-k).$$

This is clearly periodic with period 1. Thus, we can expand this into a Fourier series as:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \phi(x-k) &= \sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{2\pi i n x} \\
&= \sum_{n\in\mathbb{Z}} \left(\int_{-1/2}^{1/2} \sum_{k\in\mathbb{Z}} \phi(y-k) \mathrm{e}^{-2\pi i n y}\, \mathrm{d}y \right) \mathrm{e}^{2\pi i n x} \\
&= \sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} \int_{-1/2}^{1/2} \phi(y-k) \mathrm{e}^{-2\pi i n (y-k)}\, \mathrm{d}y \right) \mathrm{e}^{2\pi i n x} \\
&= \sum_{n\in\mathbb{Z}} \left(\int_{-\infty}^{\infty} \phi(y) \mathrm{e}^{-2\pi i n y}\, \mathrm{d}y \right) \mathrm{e}^{2\pi i n x} \\
&= \sum_{n\in\mathbb{Z}} \widehat{\phi}(n) \mathrm{e}^{2\pi i n x}.
\end{aligned}$$

Setting $x = 0$ in both sides, we get the Poisson summation formula:

$$\sum_{k\in\mathbb{Z}} \phi(k) = \sum_{k\in\mathbb{Z}} \widehat{\phi}(k).$$

Once we establish (1), i.e., $\widehat{\mathrm{III}}_1(\xi) = \mathrm{III}_1(\xi)$, it is easy to get (2) by the dilation formula. First, it is important to notice that $\delta(ax) = (1/a)\delta(x)$ or $a\delta(ax) = \delta(x)$. Now,

$$\mathrm{III}_A(x) = \sum_{k\in\mathbb{Z}} \delta(x - kA) = \sum_{k\in\mathbb{Z}} \frac{1}{A}\delta(x/A - k) = \frac{1}{\sqrt{A}} \boldsymbol{\delta}_A \,\mathrm{III}(x).$$

$$\begin{aligned}
\mathcal{F}\{\mathrm{III}_A\}(\xi) &= \mathcal{F}\left\{ \frac{1}{\sqrt{A}} \boldsymbol{\delta}_A \,\mathrm{III} \right\}(\xi) \\
&= \frac{1}{\sqrt{A}} \boldsymbol{\delta}_{1/A} \,\mathrm{III}(\xi) \\
&= \mathrm{III}(A\xi) \\
&= \sum_{k\in\mathbb{Z}} \delta(A\xi - k) \\
&= \sum_{k\in\mathbb{Z}} \frac{1}{A}\delta(\xi - k/A) \\
&= \frac{1}{A} \mathrm{III}_{1/A}(\xi).
\end{aligned}$$

Problem 2: This was an easy problem. Most of you answered correctly.

Problems 3–4: **(a)** A few people didn't simplify $\cos(\pi k)$ as $(-1)^k$. Since k is an integer, you should use $(-1)^k$. By the same token, $\sin(\pi k) = 0$, of course.

(b) The point of this problem is to figure out the difference between the hand-derived Fourier series coefficients and the the DFT coefficients computed via the FFT function of matlab. Many people plotted and compared the absolute values of the FFT coefficients and hand-computed Fourier coefficients. That only gives you a part of the story. You really need to plot and compare the real part and imaginary part separately without taking the absolute values in order to see the real difference between the hand-computed Fourier coefficients and the output of the matlab FFT function!

First of all, you need to go back to the original definition of the DFT and the Fourier coefficients.

$$
\begin{aligned}
c_k &= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\mathrm{e}^{-2\pi\mathrm{i}kx}\,\mathrm{d}x \\
&\approx \sum_{\ell=0}^{N-1} f(x_\ell)\mathrm{e}^{-2\pi\mathrm{i}kx_\ell}\Delta x, \quad x_\ell = -\frac{1}{2} + \ell\Delta x \text{ and } \Delta x = \frac{1}{N} \\
&= \frac{1}{N}\sum_{\ell=0}^{N-1} f\left(-\frac{1}{2}+\frac{\ell}{N}\right)\mathrm{e}^{-2\pi\mathrm{i}k(-1/2+\ell/N)} \\
&= \frac{\mathrm{e}^{-\pi\mathrm{i}k}}{N}\sum_{\ell=0}^{N-1} f\left(-\frac{1}{2}+\frac{\ell}{N}\right)\mathrm{e}^{-2\pi\mathrm{i}k\ell/N} \\
&= \frac{(-1)^k}{N}\sum_{\ell=0}^{N-1} f\left(-\frac{1}{2}+\frac{\ell}{N}\right)\mathrm{e}^{-2\pi\mathrm{i}k\ell/N} \\
&= \frac{(-1)^k}{N}\sum_{\ell=0}^{N-1} f_\ell\,\mathrm{e}^{-2\pi\mathrm{i}k\ell/N}.
\end{aligned}
$$

And finally, the summation portion can be computed by `fft` (non-unitary original version) in matlab. Here is my matlab script for Problem 3. I also put my codes online, so please download them and run them to see how much these two sets of coefficients agree.

```
% Problem 3

% define the basic parameters.
N=1024;
a=-0.5;
b=0.5;

% Create an array of N equidistant points over [a,b].
x=linspace(a,b,N);
```

```
% Create a function.
y = x; % In problem 4, this should be y=x.^2, of course.

% Normalize the function to have a unit L^2 norm.
ey = norm(y);
y = y/ey;

% Do the fft to approximate the Fourier series coefficients over this
% interval.  Note that we need to have 1/N here.  You need to go back
% to the original definition of the Fourier coefficients and its
% approximation by the trapezoidal rule.
% Note that fft essentially view the input data is defined over the
% interval on [0,1], instead of [-1/2,1/2].  So you need to do either
% of the following two:
% 1) Apply fftshift to the input vector before taking fft; or
% 2) Apply the complex exponential factor exp(pi*k)=(-1)^k to the output
%   of fft, which is equivalent to changing the signum of the fft results
%   alternatively as fy(2:2:N)=-fy(2:2:N), where fy=fft(y)/N.

fy = fft(y)/N;
fy(2:2:N)=-fy(2:2:N);

% Now, prepare the analytical Fourier coefficients you derived by
% hand.

c = zeros(1,N);
% c(1) = 1/12.0; % for Problem 4.
for k=1:N-1
  c(k+1)=i*(-1)^k/(2*pi*k); % c(k+1)=(-1)^k/(2*(pi*k)^2); % for Problem 4.
end


% Normalize the coefficients
c = c/ey;

% Now plot real and imaginary part separately using the semilog plot.
figure(1)
clf;
subplot(1,2,1);
plot(real(c(1:N/2)));
hold on
plot(real(fy(1:N/2)),'r.');
title('Real Part')
hold off
```

```
subplot(1,2,2);
plot(imag(c(1:N/2)));
hold on
plot(imag(fy(1:N/2)),'r.');
title('Imaginary Part')
hold off

% Let's look at the more details around the origin.
figure(2)
clf;
subplot(1,2,1);
plot(real(c(1:N/16)),'o');
hold on
plot(real(fy(1:N/16)),'r.');
title('Real Part')
hold off

subplot(1,2,2);
plot(imag(c(1:N/16)),'o');
hold on
plot(imag(fy(1:N/16)),'r.');
title('Imaginary Part')
hold off
```

Do the similar computation for Problem 4. You can see that they match closely, but not exact due to the approximation error by the trapezoidal rule and sampling. What happens if we increase the number of samples, e.g., to $N = 2^{15}$?

Problem 5: You can use the same strategy here as above. However, one thing which is very different is the comparison with the Fourier transform of the Gaussian, which is:

$$\hat{g}(\xi;\sigma) = \mathrm{e}^{-2\pi^2\sigma^2\xi^2} = \int_{-\infty}^{\infty} g(x;\sigma)\mathrm{e}^{-2\pi \mathrm{i}\xi x}\,\mathrm{d}x.$$

As in Problems 3 and 4, using the matlab `fft` function, we can only approximate the Fourier coefficients of the periodized Gaussian (or mutilated Gaussian) on $[-1/2, 1/2]$:

$$c_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} g(x;\sigma)\mathrm{e}^{-2\pi \mathrm{i}kx}\,\mathrm{d}x.$$

Actual output is even different from this c_k due to the error by the trapezoidal formula. Therefore, there are two errors involved here: 1) Truncation of the interval; and 2) error due to the trapezoidal rule. For more details, I strongly recommend to read [1, Chap. 6].

Problem 6: Most of the people got this problem right. You just need to use some trigonometric identity and the summation formula of a geometric series.

References

[1] W. L. Briggs and V. E. Henson, *The DFT: An Owner's Manual for the Discrete Fourier Transform*, SIAM, Philadelphia, PA, 1995.