MAT 280: Applied & Computational Harmonic Analysis
Homework 3: due Wednesday, 06/02/04

Problem 1: An n-dimensional *spike process* simply generates the standard basis vectors $\{\boldsymbol{e}_j\}_{j=1}^n \subset \mathbb{R}^n$ in a random order with equal probability, where $\boldsymbol{e}_j$ has one at the jth entry and all the other entries are zero. One can view this process as a unit impulse located at a random position between 1 and n.

(a) Compute the covariance matrix of this process.

(b) Show that the Karhunen-Loève basis of this process is any orthonormal basis in $\mathbb{R}^n$ containing a "DC" basis vector $\frac{1}{\sqrt{n}}(1, 1, \cdots, 1)^T$.

Problem 2: Consider the following stochastic process called the *ramp process*:

$$X(t) = t - H(t - \tau), \quad t \in [0, 1), \, \tau \sim \text{unif}[0, 1),$$

where $H(\cdot)$ is the Heaviside step function, i.e., $H(t) = 1$ if $t \geq 0$; and $H(t) = 0$ if $t < 0$.

(a) Show that the covariance function of this process is

$$\Gamma(s, t) = \min(s, t) - st, \quad 0 \leq s, t \leq 1.$$

(b) Show that the Karhunen-Loève basis functions of this process are of the following form:

$$\phi_k(t) = \sqrt{2} \sin \pi k t, \quad k = 1, 2, \ldots$$

(c) Discretize this process as follows. Let our sampling points be $t_k = \frac{2k+1}{2n}$, $k = 0, \ldots, n-1$. Suppose the discontinuity $t = \tau$ does not happen exactly at the sampling points. Then all the realizations whose discontinuities are located anywhere in the open interval $(\frac{2k-1}{2n}, \frac{2k+1}{2n})$ have the same discretized version. Therefore, any realization now has the following form:

$$\boldsymbol{x}_j = (x_{0j}, \ldots, x_{n-1,j})^T, \quad x_{kj} = \begin{cases} \frac{2k+1}{2n}, & \text{for } k = 0, \ldots, j-1, \\ \frac{2k+1}{2n} - 1, & \text{for } k = j, \ldots, n-1, \end{cases}$$

where j is picked uniformly randomly from the set $\{0, 1, \cdots, n-1\}$. (Note that the index of the vector components starts with 0 for convenience). Take $n = 256$, and generate 256 realizations of this process. (You only have to construct a data matrix X whose column vectors are $\boldsymbol{x}_j$, $j = 0, \ldots, 255$. Then compute the covariance matrix, compute the eigenvectors (i.e., KL vectors) using matlab, and compare those eigenvectors with the sinusoids analytically obtained in (b).

Problem 3: Let $\Phi \in L^2(\mathbb{R}^2)$. Then, there exists $f \in L^2(\mathbb{R})$ such that $\Phi(x,\xi) = Sf(x,\xi)$ (Sf is the windowed Fourier transform of f) if and only if

$$\Phi(x_0,\xi_0) = \int\int \Phi(x,\xi)K(x_0,x,\xi_0,\xi)\,\mathrm{d}x\,\mathrm{d}\xi,$$

where

$$K(x_0,x,\xi_0,\xi) = \langle g_{x,\xi}, g_{x_0,\xi_0}\rangle = \int g(y-x)g(y-x_0)\mathrm{e}^{-2\pi\mathrm{i}(\xi_0-\xi)y}\,\mathrm{d}y.$$

Hint: To prove the necessity, in the definition of the windowed Fourier transform of f, replace f by the reconstruction formula of f from its windowed Fourier transform. To prove the sufficiency, define f as

$$f(x) = \int\int \Phi(y,\xi)g(x-y)\mathrm{e}^{2\pi\mathrm{i}\xi x}\,\mathrm{d}\xi\,\mathrm{d}y,$$

and show $\Phi(x,\xi) = Sf(x,\xi)$.

Problem 4: Prove that if $K \in \mathbb{R} - \{0\}$, then the set $\{\phi_k(x) = \exp(2\pi\mathrm{i}kx/K)\}_{k\in\mathbb{Z}}$ forms a *tight frame* of $L^2[0,1]$. Then, compute the frame bounds.