Problem 1: Prove that the Fourier transform maps $L^1(\mathbb{R})$ into $BC(\mathbb{R})$, the space of bounded continuous functions on $\mathbb{R}$.

Problem 2: Prove that the dilation operator:
\[
\delta_s f(x) \overset{\Delta}{=} \frac{1}{\sqrt{s}} f \left( \frac{x}{s} \right), \quad s > 0,
\]
is an isometry (i.e., norm-preserving) in $L^2(\mathbb{R})$.

Problem 3: Suppose $f, g \in L^1$. Prove the following Fourier transform formulas:

(a) $\mathcal{F}\{\tau_\alpha f\}(\xi) = e^{-2\pi i \alpha \xi} \hat{f}(\xi)$, where $\alpha \in \mathbb{R}$.

(b) $\mathcal{F}\{\delta_s f\}(\xi) = \delta_{1/s} \hat{f}(\xi) = \sqrt{s} \hat{f}(s\xi)$, where $s > 0$.

(c) If $f \in C^1(\mathbb{R})$ and $f'(x) \to 0$ as $|x| \to \infty$, then $\mathcal{F}\{f'\}(\xi) = (2\pi i \xi) \hat{f}(\xi)$.

(d) $\mathcal{F}\{f \ast g\}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$.

Problem 4: Compute the Fourier transform of the Gaussian function:
\[
g(x; \sigma) = \frac{1}{\sqrt{2\pi \sigma}} e^{-x^2/2\sigma^2}, \quad x \in \mathbb{R}, \sigma > 0.
\]

Problem 5: Prove that the equality in the Heisenberg inequality for $f \in L^2(\mathbb{R})$,
\[
\Delta_{x_0} f \Delta_{\xi_0} \hat{f} \geq \frac{1}{16\pi^2},
\]
with $x_0 = \xi_0 = 0$ is satisfied if and only if $f(x) = a \exp(-x^2/2\sigma^2)$ for some constants $a, \sigma \in \mathbb{R}$.

[Hint: Recall when the equality happens for the Cauchy-Schwarz inequality.]