MAT 271: Computational Harmonic Analysis
Comments on Homework 1

Problem 1: There are a couple of ways to show $\hat{f}(\xi)$ is continuous. One is to use the Dominant Convergence Theorem (D.C.T.) as follows. Because $|f(x)\mathrm{e}^{-2\pi\mathrm{i}(\xi+h)x} - f(x)\mathrm{e}^{-2\pi\mathrm{i}\xi x}| \leq 2|f(x)|$ and $f \in L^1(\mathbb{R})$, the D.C.T. implies that $\hat{f}(\xi+h) \to \hat{f}(\xi)$ as $h \to 0$. The other way is to split the integral into the two regions of integration:

$$\begin{aligned} |\hat{f}(\xi+h) - \hat{f}(\xi)| &\leq \int_{\mathbb{R}} |\mathrm{e}^{-2\pi\mathrm{i}hx} - 1||f(x)|\,\mathrm{d}x \\ &= \left(\int_{|x|\leq M} + \int_{|x|\leq M}\right) |\mathrm{e}^{-2\pi\mathrm{i}hx} - 1||f(x)|\,\mathrm{d}x \end{aligned}$$

Given $\epsilon > 0$, the second integral can be made less than ϵ by taking M sufficiently large. The first integral is majorized by

$$2\pi|h| \int_{|x|\leq M} |x||f(x)|\,\mathrm{d}x.$$

Therefore, with this choice of M, we have

$$\limsup_{h\to 0} \sup_{\xi\in\mathbb{R}} |\hat{f}(\xi+h) - \hat{f}(\xi)| \leq \epsilon.$$

But ϵ was arbitrary. This means that in fact, $\hat{f}(\xi)$ is *uniformly* continuous. This way of proving the uniform continuity can be easily generalized to n-dimensional Fourier transforms.

Problem 2: Caution: Isometry means that $\|\delta_s f\|_2 = \|f\|_2$ in the L^2 norm.

Problem 3: (c) You need to justify that $f(x) \to 0$ as $|x| \to \infty$. (d) You need to justify why the order of integrations can be swapped.

Problem 4: There are several ways to derive the Fourier transform of the Gaussian. I believe the best way is the following.

Consider the derivative:

$$\begin{aligned} &g'(x;\sigma) = -\frac{1}{\sqrt{2\pi}\sigma^3} x\mathrm{e}^{-x^2/2\sigma^2}. \\ &\Longrightarrow \sigma^2 g' = -xg \\ &\Longrightarrow \sigma^2(2\pi\mathrm{i}\xi)\hat{g} = -\frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}\hat{g}}{\mathrm{d}\xi} \\ &\Longrightarrow \frac{\mathrm{d}\hat{g}}{\mathrm{d}\xi} = -4\pi^2\sigma^2\xi\hat{g} \end{aligned}$$

This is a simple ODE and we can get the solution:

$$\hat{g}(\xi;\sigma) = C\mathrm{e}^{-2\pi^2\sigma^2\xi^2}.$$

But $\hat{g}(0) = 1$ because this is the integral of the probability density function of the normal distribution with mean 0 and variance σ^2. Therefore, $C = 1$.

$$\hat{g}(\xi;\sigma) = \mathrm{e}^{-2\pi^2\sigma^2\xi^2}.$$

Problem 5: If you state the equality condition of the Cauchy-Schwarz inequality used in this uncertainty inequality, then it is automatically, "if and only if". The bottom line is the Cauchy-Schwarz inequality in this case becomes:

$$\|f\|^4 = 4\left(\mathrm{Re}\int x\overline{f(x)}f'(x)\,\mathrm{d}x\right)^2 \leq \int x^2|f(x)|^2\,\mathrm{d}x \int |f'(x)|^2\,\mathrm{d}x,$$

and the equality holds *if and only if*

$$f'(x) = cxf(x), \quad \text{for some constant } c.$$

So, we can easily get the solution:

$$f(x) = a\mathrm{e}^{cx^2/2}, \quad \text{for some constants } a, c.$$

However, the function f must be in $L^2(\mathbb{R})$. So, we must have $c < 0$. Otherwise, this function cannot have a finite norm in $L^2(\mathbb{R})$. So, we can set $c = -1/\sigma^2$ for some $\sigma > 0$, and get the form:

$$f(x) = a\mathrm{e}^{-x^2/2\sigma^2}, \quad \text{for some constants } a \text{ and } \sigma > 0.$$