MAT 271: Applied & Computational Harmonic Analysis
Homework 2: due Monday, 05/03/06

Problem 0: Familiarize yourself to the MATLAB environment using the MATLAB primers. (Only applicable for the people who do not have much MATLAB experience.) See "Useful Link" in my course web page to get MATLAB primers.

Problem 1: Let $\mathrm{III}_A(x) := \sum_{k\in\mathbb{Z}} \delta(x - kA)$ be the *Shah* function with period A. Prove:

$$\mathcal{F}\{\mathrm{III}_A\}(\xi) = \frac{1}{A}\,\mathrm{III}_{1/A}(\xi) = \frac{1}{A}\sum_{k\in\mathbb{Z}} \delta(\xi - k/A),$$

where $\delta(\cdot)$ is the Dirac delta function.

Problem 2: Let $\boldsymbol{w}_N^k := \frac{1}{\sqrt{N}}\left[\omega_N^0, \omega_N^k, \omega_N^{2k} \ldots, \omega_N^{k(N-1)}\right]^T \in \mathbb{C}^N$, where $\omega_N = \exp(2\pi\mathrm{i}/N)$. Prove

$$\langle \boldsymbol{w}_N^k, \boldsymbol{w}_N^\ell \rangle = \delta_{k,\ell},$$

where $\delta_{k,\ell}$ is Kronecker's delta.

Problem 3: Consider a periodized versions of the function over $[-1/2, 1/2]$:

$$f(x) = ax, \quad -\frac{1}{2} \le x < \frac{1}{2}, \quad a > 0.$$

(a) Compute the Fourier coefficients c_k of this periodic function by hand.

(b) Using MATLAB, do the following:

- **1)** Determine the value of a so that after the discretization of this function on a uniform grid of length 1024, the resulting vector has a unit ℓ^2-norm;
- **2)** Compute FFT of this vector (via `fft`);
- **3)** Display the absolute value of the Fourier coefficients;
- **4)** Plot the Fourier coefficients in (a) with a computed in 1);
- **5)** Do they agree? What is your reasoning if they do not.

Problem 4: Consider a periodized version (with period 1) of the following function:

$$f(x) = ax^2, \quad -\frac{1}{2} \le x < \frac{1}{2}, \quad a > 0.$$

Repeat (a), (b) of Problem 3 for this function. In addition,

(c) Compare the speed of the decay of the Fourier coefficients of this function with that of Problem 3. Which decays faster? Why?

Problem 5: Consider a periodized version (with period 1) of the following function:

$$f(x) = a\mathrm{e}^{-x^2/2\sigma^2}, \quad -\frac{1}{2} \le x < \frac{1}{2}, \quad a, \sigma > 0.$$

Repeat (1)–(3) of (b) of Problem 3 with $\sigma = 1, 0.1, 0.01$. In addition,

4) Compare the speed of the decay of the Fourier coefficients of this function with these different values of σ;

5) Compare these decays with those of Problems 3 and 4. Which decays faster? Why?

6) Do they agree with the discretized version of the Fourier transform formula of Problem 4 of HW #1 (with appropriate multiplicative constants)? If not, state your interpretation/reasoning.

Problem 6: Prove the following identity:

$$S_n := \frac{1}{2} + \cos\theta + \cos 2\theta + \cdots + \cos(n-1)\theta + \frac{1}{2}\cos n\theta = \frac{1}{2}\sin n\theta \cdot \cot\frac{\theta}{2}\,.$$

Problem 7: Read the paper of the Discrete Cosine Transform by Gilbert Strang, which you can download from `http://epubs.siam.org/SIREV/volume-41/art_33674.html`.